An Exact Analytic Technique for Simulating Uniform RC Lines *

Jaijeet S. Roychowdhury A. Richard Newton Donald O. Pederson

Department of Electrical Engineering and Computer Sciences
University of California, Berkeley

Abstract

A new technique, based on convolution, has been developed for the time domain simulation of uniform RC lines. In contrast to existing techniques which approximate the line's responses, this technique is exact, requiring no simplification of the line's internal mechanism. It is shown that though the impulse responses of uniform RC lines are ill-behaved and unsuitable for direct numerical implementation, the use of a convolution formula obtained by generalizing the trapezoidal integration method leads to well-behaved analytic forms that can be directly implemented. Although the new technique involves more computation than segmentation based techniques, it makes no approximation to the uniform distribution of resistance and capacitance. Experimental results using industrial IC interconnect demonstrate the efficacy of the new technique.

1 Introduction

The accurate simulation of uniform RC lines is important in the design and verification of VLSI circuits, in which the interconnections are usually modelled to have uniformly distributed resistance and capacitance. RC charging effects are primarily responsible for gate-to-gate delays in digital ICs, and with the continuous improvement in the intrinsic switching speeds of semiconductor devices, these delays are becoming the primary factor limiting system performance.

So far, all existing methods for the transient simulation of uniform RC lines have, to our knowledge, involved reduced order models (lumped approximations) approximate the uniform distribution of resistance and capacitance (see Section 2 for a brief review). In this paper, an *exact analytic formulation* for transient simulation is presented. The method is based on convolving the impulse responses of the uniform RC line with the voltages and currents applied to its ends[1]. The novelty and exactness of the new technique stem from the fact that although the impulse responses are shown to be themselves unsuitable for numerical implementation (for they possess singularities at zero), the use of a technique for numerical convolution [1] leads to functions that are not only well-behaved but are also expressible in analytic closed form. It is to be noted that the analytic forms are not a special case of the RLC formulation [1], for which the non-zero nature of the inductance L is a fundamental requirement. In addition to removing the need for dealing with the ill-behaved impulse responses, the formulation provides $O(h^2)$ accuracy for time-step h. Experimental results demonstrate that the accuracy of the analytic technique compares favourably with uniform segmentation.

In Section 2, previous work is briefly reviewed. In Section 3, the analytic technique is presented. In Section 4, experimental results for industrial circuits are presented, followed by comments in Section 5.

*This research was supported by AT&T Bell Laboratories, IBM, Raytheon, and the California State MICRO program.

[1]Convolution has been applied previously to simulating RLC transmission lines; see Section 2.

2 Previous Work

A well-known set of methods for the simulation of uniform RC lines is that of dividing the line into a number of segments and representing each segment using lumped resistors and capacitors. Variants range from single-section to nonuniform multiple-section models. Rajput [2] has proposed a single-section model based on an approximation of the first three terms of the voltage transfer function and the open-circuit input impedance of the uniform RC line. This model is asymmetric with respect to the two ends, however, and its range of validity is limited in frequency. Gupta et al [3] proposed a different single section model based on an approximation of the first two terms of the transfer function, while using an exact expression for other terms. This model, while being symmetric, is still limited in frequency and contains a negative capacitance that can lead to stability problems.

The uniform multiple-section model, in which the line is split into a number of uniform segments, is the most commonly used model today. Asymmetric versions of this model are widely used, and symmetric versions are also available [4, 5]. This technique has the advantage of conceptual simplicity and convenience, since no change needs to be made to an existing simulator for it to be able to accept the model. On the other hand, the number of segments needed for adequate accuracy can vary from application to application. Gopal et al [6] have developed a formula for the optimal number of segments to use, using moment-matching methods. An estimate of the approximate bandwidth of the signals in the circuit is required as input to this formula.

A nonuniform segmentation model, proposed by Gertzberg [7], has the advantage of requiring fewer sections than uniform segmentation for a given accuracy. [7] also contains a formula to estimate the number of sections to use. This formula, however, requires the use of a nonuniformity constant K, which again needs to be decided by experimentation or prior experience.

The present work exploits the well-known convolution property of all linear systems (the uniform RC line being linear), i.e., the outputs of a linear system can be obtained by convolving the inputs with functions characteristic to the linear system, called impulse responses. Djordjević et al were apparently first to use this method [8] in connection with circuit simulation, applying it to RLC transmission lines. In previous work [1], the present authors extended the convolution technique by proposing an improved numerical convolution formula and by identifying analytic forms for the impulse responses of RLC lines. As brought out below, the results of the present work are not a direct consequence of the RLC line formulation.

3 Analytic Formulation

The transient behaviour of a uniform RC line with resistance R and capacitance C per unit length is described by the following partial differential equations (the Telegrapher Equations [9] with zero inductance and parallel conductance):

$$\frac{\partial v}{\partial x} = -R\, i \quad (1)$$

$$\frac{\partial i}{\partial x} = -C\, \frac{\partial v}{\partial t} \quad (2)$$

The above equations hold for x varying between 0 and l, the

0-8186-2780-8/92 $3.00 © 1992 IEEE

length of the uniform RC line. $v(x,t)$ and $i(x,t)$ are the voltage and current at the point x in the line at time t, respectively. It is assumed that the simulation starts from time 0. It is also assumed that at time $t=0$, the circuit is at its quiescent, or DC, state.

The inputs to the transmission line are the port variables $v_1(t) = v(0,t)$, $i_1(t) = i(0,t)$, $v_2(t) = v(l,t)$ and $i_2(t) = -i(l,t)$. These four port variables specify the *boundary conditions* of Equations 1 and 2.

Laplace transforms are taken (in t) of Equations 1 and 2 to arrive at ordinary differential equations in x and s, the Laplace variable:

$$\frac{\partial V}{\partial x} = -RI \tag{3}$$

$$\frac{\partial I}{\partial x} = -sCV \tag{4}$$

V, I refer to $V(x,s)$ and $I(x,s)$, the Laplace transforms of $v(x,t)$ and $i(x,t)$.

To uncouple the above equations, a basis change is performed from the variables V and I to new ("scattering parameter") variables p and q, defined as follows:

$$p(x,s) = \frac{V(x,s) + Z(s)I(x,s)}{2} \tag{5}$$

$$q(x,s) = \frac{V(x,s) - Z(s)I(x,s)}{2} \tag{6}$$

$Z(s)$ is the frequency-domain characteristic impedance of the line:

$$Z(s) = \sqrt{\frac{R}{sC}} \tag{7}$$

Equations 5 and 6 are rewritten to express V and I in terms of p and q; using these, Equations 3 and 4 are rewritten in terms of p and q. Two decoupled linear first-order ODEs in x are obtained by adding and subtracting Equations 3 and 4:

$$\frac{\partial p}{\partial x} + \lambda(s)p = 0 \tag{8}$$

$$\frac{\partial q}{\partial x} - \lambda(s)q = 0 \tag{9}$$

$\lambda(s)$ is the frequency-domain propagation constant of the line:

$$\lambda(s) = \sqrt{sCR} \tag{10}$$

The general solution of any first-order ordinary differential equation of the type

$$\frac{\partial y}{\partial x} + P(x)y = 0 \tag{11}$$

is

$$y = C_1 e^{-\int P(x)\,dx} \tag{12}$$

Equation 12 is applied to Equations 8 and 9 to obtain the solutions for p and q (abbreviating $\lambda(s)$ by λ):

$$p(x,s) = Ae^{-\lambda x} \tag{13}$$

$$q(x,s) = Be^{\lambda x} \tag{14}$$

The boundary condition at $x=0$ is applied to Equation 13, and that at $x=l$ to Equation 14, to determine the constants A and B. These are substituted into Equations 13 and 14 to obtain:

$$p(x,s) - e^{-\lambda x}p(0,s) = 0 \tag{15}$$

$$q(l,s)e^{-\lambda(l-x)} - q(x,s) = 0 \tag{16}$$

Now p and q are written in terms of V and I (using Equations 5 and 7), and the resulting equations divided by $Z(s)$ to obtain:

$$\left[V(x,s)Y(s) + I(x,s)\right] - e^{-\lambda x}\left[V(0,s)Y(s) + I(0,s)\right] = 0 \tag{17}$$

$$\left[V(l,s)Y(s) - I(l,s)\right]e^{-\lambda(l-x)} - \left[V(x,s)Y(s) - I(x,s)\right] = 0 \tag{18}$$

where

$$Y(s) = \frac{1}{Z(s)} = \sqrt{\frac{sC}{R}} \tag{19}$$

By substituting $x=l$ in Equation 17 and $x=0$ in Equation 18, and noting that $V(0,s) = V_1(s)$, $I(0,s) = I_1(s)$, $V(l,s) = V_2(s)$ and $I(l,s) = I_2(s)$, where V_1, V_2, I_1 and I_2 denote the Laplace transforms of the port variables $v_1(t)$, $v_2(t)$, $i_1(t)$ and $i_2(t)$, the following frequency domain constitutive equations are obtained:

$$\left[V_2(s)Y(s) + I_2(s)\right] - e^{-\lambda l}\left[V_1(s)Y(s) + I_1(s)\right] = 0 \tag{20}$$

$$\left[V_2(s)Y(s) - I_2(s)\right]e^{-\lambda l} - \left[V_1(s)Y(s) - I_1(s)\right] = 0 \tag{21}$$

Let h_Y, h_λ, and $h_{\lambda Y}$ denote the inverse Laplace transforms of $Y(s)$, $e^{-\lambda(s)l}$, and $Y(s)e^{-\lambda(s)l}$ respectively. Equations 20 and 21 are Laplace inverted to obtain the time domain formulation:

$$\left[v_2(t)*h_Y(t) - i_2(t)\right] - \left[v_1(t)*h_{\lambda Y}(t) + i_1(t)*h_\lambda(t)\right] = 0 \tag{22}$$

$$\left[v_2(t)*h_{\lambda Y}(t) + i_2(t)*h_\lambda(t)\right] - \left[v_1(t)*h_Y(t) - i_1(t)\right] = 0 \tag{23}$$

In Equations 22 and 23, $*$ denotes the convolution operator (see next paragraph for a definition). h_Y, h_λ and $h_{\lambda Y}$ are the three impulse responses of the uniform RC line. It is shown later in this section that h_Y possesses a singularity at zero. Next, convolution of the port variables v_1, v_2, i_1 and i_2 with these is considered.

The convolution operation between two functions is defined as follows:

$$x(t)*h(t) = \int_0^t x(\tau)\,h(t-\tau)\,d\tau \tag{24}$$

In Equation 24, $x(t)$ represents any of the port variables v_1, v_2, i_1 and i_2, while $h(t)$ represents any of the impulse responses h_Y, h_λ, $h_{\lambda Y}$. In order to perform the above computation numerically in a simulator, one is constrained to dealing with samples of $x(t)$ at discrete points on the time axis. Assume that the simulator is at the time-point t_n, having already stepped from $t_0 = 0$ through t_2, $t_3, \ldots, t_{n-1}$. Denote the samples of $x(t)$ at $t_0, \ldots, t_n$ by $x_0, \ldots, x_n$.

In previous work [1], a numerical formula was developed that computes Equation 24 from the samples x_i, using a function related to $h(t)$. The formula, reproduced below, is a generalization of the well-known trapezoidal integration technique for ordinary differential equations and is exact if $x(t)$ is piecewise linear between its sample points.

$$\int_0^{t_n} x(\tau)\,h(t_n-\tau)\,d\tau \approx x_n\frac{F(h,t_n-t_{n-1})}{t_n-t_{n-1}} + \sum_{i=1}^{n-1} x_i\left[\frac{F(h,t_n-t_{i-1})-F(h,t_n-t_i)}{t_i-t_{i-1}} - \frac{F(h,t_n-t_i)-F(h,t_n-t_{i+1})}{t_{i+1}-t_i}\right] \tag{25}$$

In Equation 25, $F(h,t)$ is defined as follows:

$$F(h,t) \triangleq \int_0^t \int_0^T h(\tau')\, d\tau'\, d\tau \tag{26}$$

Note that the first argument of $F(\cdot,\cdot)$ in Equation 26 is a function and not a single real number. If the assumption that $x(t)$ is piecewise linear is not strictly valid, the integration formula in Equation 25 has an error term. It can be shown [10] that this error is proportional to the second derivative of $x(t)$.

In Equation 25, it is to be noted that nowhere is the impulse response $h(t)$ itself used, so long as its twice repeated integral, $F(h,t)$, is available. It is shown next that analytic expressions exist for $F(h_Y,t)$, $F(h_\lambda,t)$ and $F(h_{\lambda Y},t)$. In doing so, the following elementary property of Laplace transforms is used (let $\mathcal{L}$ denote the Laplace transform operator):

$$\mathcal{L}\{F(h,t)\} = \mathcal{L}\left\{\int_0^t \int_0^T h(\tau')\, d\tau'\, d\tau\right\} = \frac{\mathcal{L}\{h(t)\}}{s^2} \tag{27}$$

Using Equation 27, note that:

$$\mathcal{L}\{F(h_Y,t)\} = \sqrt{\frac{C}{R}}\, s^{-\frac{3}{2}} \tag{28}$$

$$\mathcal{L}\{F(h_\lambda,t)\} = \frac{e^{-\sqrt{sRCl^2}}}{s^2} \tag{29}$$

$$\mathcal{L}\{F(h_{\lambda Y},t)\} = \sqrt{\frac{C}{R}}\, \frac{e^{-\sqrt{sRCl^2}}}{s^{\frac{3}{2}}} \tag{30}$$

No.	$F(s)$	$f(t)$
1	$\frac{1}{s^{\frac{3}{2}}}$	$\frac{2}{\sqrt{\pi}}\sqrt{t}$
2	$\frac{1}{s^{\frac{3}{2}}} e^{-\sqrt{sT}}$	$2\sqrt{\frac{t}{\pi}}\, e^{-\frac{T}{4t}} - \sqrt{T}\, erfc\left(\frac{\sqrt{T}}{2\sqrt{t}}\right)$
3	$\frac{1}{s^2} e^{-\sqrt{sT}}$	$\left(\frac{T}{2}+t\right) erfc\left(\frac{\sqrt{T}}{2\sqrt{t}}\right) - \sqrt{\frac{Tt}{\pi}}\, e^{-\frac{T}{4t}}$

Table 1: Laplace Transform Pairs (Fodor[11])

From Table 1, expressions are obtained for $F(h_Y,t)$, $F(h_\lambda,t)$, and $F(h_{\lambda Y},t)$:

$$F(h_Y,t) = 2\sqrt{t\frac{C}{\pi R}} \tag{31}$$

$$F(h_\lambda,t) = (t + RCl^2)\, erfc\left(\sqrt{\frac{RCl^2}{4t}}\right) - \sqrt{\frac{RCl^2}{\pi}}\, t e^{-\frac{RCl^2}{4t}} \tag{32}$$

$$F(h_{\lambda Y},t) = \sqrt{\frac{C}{R}}\left[2\sqrt{\frac{t}{\pi}}\, e^{-\frac{RCl^2}{4t}} - \sqrt{RCl^2}\, erfc\left(\sqrt{\frac{RCl^2}{4t}}\right)\right] \tag{33}$$

In the above, $erfc(\cdot)$ denotes the complementary error function [12], defined by:

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt \tag{34}$$

Using the expressions for $F(\cdot,\cdot)$ from Equations 31, 32 and 33 in Equation 25, the convolution operations in Equations 22 and 23 can be implemented numerically.

In order to obtain expressions for h_Y, h_λ, and $h_{\lambda Y}$, the corresponding expressions for $F(\cdot,t)$ need only be differentiated twice. For $h_Y(t)$, the expression obtained is immediately seen to have a singularity at zero:

$$h_Y(t) = F''(h_Y,t) = -\frac{1}{2}\sqrt{\frac{C}{\pi R t^3}} \tag{35}$$

Thus it is impossible to use the values of h_Y directly to compute the convolution; instead, the above formulation using $F(h_Y,t)$ must be used.

4 Experimental Results

The formulae of the preceding section were implemented in an experimental version of the circuit simulator SPICE version 3e1. Results on two practical circuits are presented in this section.

On-chip interconnect parameters for the first circuit, denoted by **ibm**, were taken from the paper by Cottrell et al [13]. The interconnect uses 1μm technology of thickness about 1μm. For aluminium interconnect, the resistance was calculated to be 20m-ohm μm^{-1} or 20ohm mm^{-1}. The capacitance, taken from [13], was 0.3fF μm^{-1} or 0.3pF mm^{-1}. The length of the line was assumed to be the edge dimension of a large chip, 1cm. The circuit consists of a logic gate (inverter) driving a uniform RC line left open at the far end.

Fig. 1 displays the waveforms at the receiving end of the line. Waveforms produced by using the analytic technique of this paper, as well as by simple segmentation using 1, 2, 3, 4, 5 and 10 segments, are shown. Fig. 2 zooms in on part of Fig. 1. The improvement in accuracy with the use of more segments, as well as the accuracy of the analytic technique, is easily seen. Execution times for segmentation using 1, 2, 3, 4, 5 and 10 segments were 1.79, 1.99, 2.04, 2.07, 1.97 and 2.04 seconds, respectively, on a DEC 5500 computer running ULTRIX 4.2. The execution time using the analytic technique was 2.18 seconds.

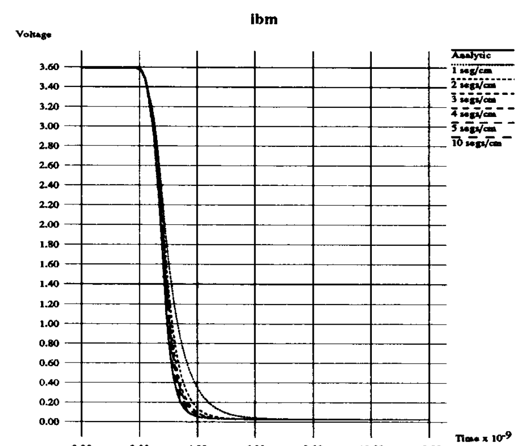

Figure 1: Far-end voltage, **ibm**

The second circuit (denoted by **ucb**) uses 3μm technology from [14]. This translates to a capacitance value of 1fF μm^{-1} or 1pF mm^{-1}, and a resistance value (assuming Al) of 6.6667m-ohm μm^{-1} or 6.6667ohm mm^{-1}. The length was again assumed to be 1cm. The circuit in this case consists of a voltage source driving a nonlinear diode load through the uniform RC line.

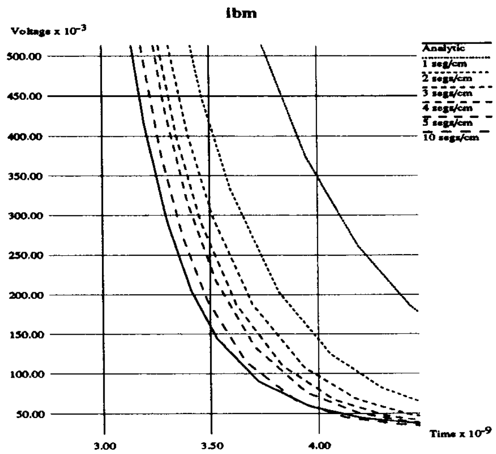

Figure 2: Far-end voltage (detail), **ibm**

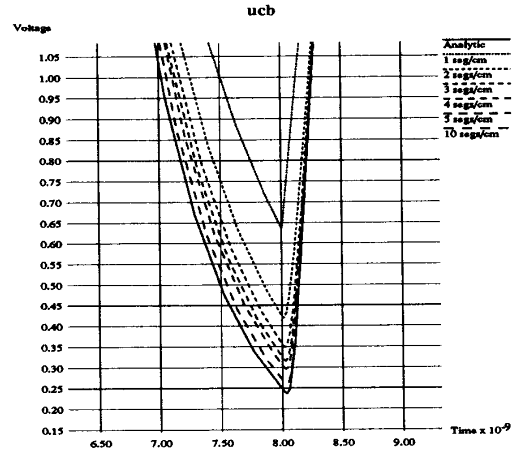

Figure 4: Load voltage (detail), **ucb**

Fig. 3 displays the waveforms at the load end of the line, while Fig. 4 zooms in on Fig. 3. Execution times for segmentation using 1, 2, 3, 4, 5 and 10 segments were 0.26, 0.31, 0.33, 0.37, 0.34 and 0.42 seconds, respectively. The execution time using the analytic technique was 0.66 seconds.

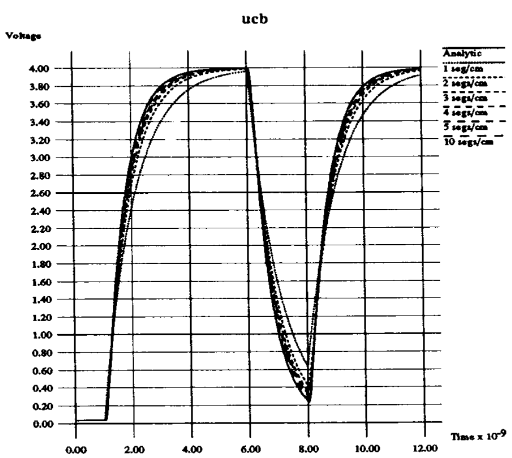

Figure 3: Load voltage, **ucb**

5 Comments

In this work, a new technique for the simulation of uniform RC lines has been formulated. It has been shown that though the impulse responses for the RC line are not well-behaved, convolution is possible through a numerical formula developed in previous work. Further, exact analytic forms more useful than the impulse responses themselves have been presented. Although more expensive computationally than uniform segmentation, the new technique makes no approximation to the uniform RC distribution, a feature lacking in previous techniques. Experimental results for practical circuits testify to the accuracy of the new technique.

Because the technique is based on convolution, it has the disadvantage of being *quadratic time*, i.e., the computation required for a simulation of duration T increases as T^2. This is a fundamental property of simulation by direct convolution. It has been shown [15] that in the case of uniform RLC lines, an alternative technique exists that is as accurate as convolution while being linear time.

Acknowledgments

Gerry Marino's encouragement and support of this work is greatly appreciated. Thanks are due to Edoardo Charbon and Umakanta Choudhury for helping with the evaluation. Administrative help from Kia Cooper is very much appreciated.

References

[1] J.S. Roychowdhury and D.O. Pederson. Efficient Transient Simulation of Lossy Interconnect. In *Proc. IEEE Design Automation Conference*, pages 740–745, San Francisco, CA, June 1991.

[2] Y.V. Rajput. Modeling Distributed RC lines for the transient analysis of complex networks. *Int. J. Electronics*, 36(5):709–717, 1974.

[3] M.L. Gupta and A.B. Bhattacharya. Single-Section Lumped Models for Integrated Circuit Resistors. *Solid State Electronics*, 16(12):1506–1509, December 1973.

[4] M.S. Ghausi and J.J. Kelly. *Introduction to distributed-parameter networks; with application to integrated circuits.* Holt, Rinehart and Winston, New York, 1968.

[5] T. Sakurai. Approximation of wiring delay in MOSFET LSI. *IEEE J. Solid-State Ckts.*, 18(4), 1983.

[6] N. Gopal, D.P. Neikirk, and L.T. Pillage. Evaluating RC-Interconnect Using Moment-Matching Approximations. In *Proc. IEEE International Conference on Computer Aided Design*, pages 74–77, November 1991.

[7] L. Gertzberg. Modeling and Applications of Solid-State Uniform Distributed RC Lines. In *Techniques and Applications of Computer-Aided Simulation for IC and System Design, Part 2.* Integrated Circuits Laboratory, Stanford University, Stanford, CA.

[8] A.R. Djordjević et al. Analysis of Lossy Transmission Lines with Arbitrary Nonlinear Terminal Networks. *IEEE Trans. Microwave Th. Tech.*, MTT-34(6):660, June 1986.

[9] J.E. Schutt-Aine and R. Mittra. Nonlinear Transient Analysis of Coupled Transmission Lines. *IEEE Trans. Ckts. Sys.*, 36(7), July 1989.

[10] J.S. Roychowdhury, A. R. Newton, and D.O. Pederson. Algorithms for the Transient Simulation of Lossy Interconnect. Submitted to IEEE Trans. CAD.

[11] G. Fodor. *Laplace Transforms in Engineering*. Akademiai Kiado, Budapest, 1965.

[12] M. Abramowitz and I.A. Stegun, editors. *Handbook of Mathematical Functions*. Dover Books on Advanced Mathematics. Dover Publicatons, Inc., New York, 1970.

[13] P.E. Cottrell and E.M. Buturia. VLSI Wiring Capacitance. *Bell Sys. Tech. J.*, 29(3):277–288, May 1985.

[14] E. Charbon, E. Malavasi, U. Choudhury, A. Casotto, and A. Sangiovanni-Vincentelli. A constraint-driven placement methodology for analog integrated circuits. In *Proc. IEEE Custom Integrated Circuits Conference*, May 1992.

[15] J.S. Roychowdhury et al. An Impulse-Response Based Linear Time-Complexity Algorithm for Lossy Interconnect Simulation. In *Proc. IEEE International Conference on Computer Aided Design*, pages 62–65, Santa Clara, CA, November 1991.