THEORY AND ALGORITHMS FOR RF SENSITIVITY COMPUTATION

Jaijeet Roychowdhury

ECE Department, University of Minnesota, Minneapolis, MN, USA

ABSTRACT

Finding performance sensitivities with respect to controllable parameters is important in RF design, both for making manual improvements and for automated optimization. In this paper, we obtain efficient sensitivity techniques for common RF analyses – harmonic balance, shooting, and envelope simulation.

1. INTRODUCTION

During circuit and system design, knowledge of the sensitivities of performance metrics to design parameters is often desirable. Knowing the impact of a small change in the circuit (*e.g.*, in the width of a transistor) on some desired functional characteristic (*e.g.*, the gain of an amplifier, or the delay of a logic gate) can guide manual improvements to the design. Sensitivity information is especially useful if automated optimization is employed to maximize performance, because gradient-based optimization techniques rely on sensitivities to guide their search. While the desired sensitivities can always be calculated by finite differencing, it is considerably more efficient and accurate to employ specialized techniques based on linearization and adjoint calculations.

In recent years, there has been renewed interest and activity in RF circuit and system design, in good measure due to the explosion in portable wireless communications. This has been accompanied by greater use of CAD tools for RF design, featuring specialized algorithms for steady-state and envelope calculation. While significant advances have been made in improving the efficiency and applicability of such algorithms, these advances do not appear to have carried over to RF sensitivity computations.

Efficient sensitivity computation for DC and small-signal circuit analysis has long been established [1]. Procedures for computing transient sensitivities using adjoint techniques have also been developed [2,3]. Sensitivity calculation techniques for harmonic balance, a steady state calculation technique useful in RF design, are also available [4–6]; however, they often limit generality to gain efficiency, and do not scale well with increasing circuit size. Sensitivity computation methods for another steady-state technique, shooting, have also been developed, particularly amongst the power systems community [7, 8]. These techniques, too, typically rely on specific circuit characteristics. Furthermore, there appears to be no work addressing sensitivity computations in the context of envelope simulation.

In this work, we present general sensitivity computation methods for three common RF analysis techniques: harmonic balance, shooting and envelope simulation. We show that recent 'fast' methods for performing these analyses [9–11] are easily employed for corresponding sensitivity calculations. In particular, we show that adjoint sensitivity calculations are especially convenient when corresponding steady-state or envelope methods use underlying Lanczos methods for iterative linear solution.

2. PRELIMINARIES

In this section, we review basic sensitivity techniques for nonlinear 'algebraic' and differential equations, focussing on adjoint methods.

2.1. Adjoint sensitivities for 'algebraic' nonlinear systems

Given a vector system of n nonlinear equations depending on n unknowns x and m parameters p, *i.e.*,

$$f(x,p) = 0, \quad f, x \in \mathcal{R}^n, p \in \mathcal{R}^m, \tag{1}$$

the total derivative is given by

$$\underbrace{\left.\frac{\partial f}{\partial x}\right|_{x,p}}_{G} \delta x + \underbrace{\left.\frac{\partial f}{\partial p}\right|_{x,p}}_{S} \delta p \sim 0, \quad G \in \mathcal{R}^{n\times n},\ S \in \mathcal{R}^{n\times m}. \tag{2}$$

Hence the sensitivity matrix $x_p \in \mathcal{R}^{n\times m}$ is given by

$$x_p = -G^{-1}S \quad \text{(assuming } G \text{ invertible)}, \tag{3}$$

and, if $\delta x = x_p\,\delta p$, δx is given by the solution of the linear equation

$$G\delta x = -S\delta p. \tag{4}$$

(4) may be solved numerically to calculate δx when specific δp are available. If, however, we are interested only in a weighted sum of the elements of δx, *i.e.*, in $y = c^T\delta x$, then it is possible to to arrive at a functional form for y in terms of δp:

$$y = -c^T G^{-1} S \delta p = -\left[G^{-T}c\right]^T S\delta p, \tag{5}$$

or, denoting $r = G^{-T}c$,

$$y = -r^T S \delta p = s^T \delta p \quad \text{where } s = -S^T r. \tag{6}$$

The advantage of (6) is that finding the *gradient vector* s involves only a *single* linear solution of the adjoint system

$$G^T r = c, \tag{7}$$

followed by the matrix-vector multiplication $s = S^T r$. It is often possible to perform these operations efficiently (for example, when G and S are sparse).

2.2. Adjoint sensitivities for differential equations

Consider a system of nonlinear differential equations, depending on a set of parameters p:

$$\dot{x} = f(x,p) + b(t), \quad x(t), b(t), f \in \mathcal{R}^n, p \in \mathcal{R}^m, \tag{8}$$

with initial condition $x(t_0) = x_0(p)$. The assumption that the forcing term $b(t)$ is independent of p is made without loss of generality for convenience. We assume that (8) satisfies conditions for existence and uniqueness of solutions. The total derivative of (8) is given by

$$\frac{\partial \delta x}{\partial t}(t) = \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x(t),p}}_{G(t)} \delta x + \underbrace{\left.\frac{\partial f}{\partial p}\right|_{x(t),p}}_{S(t)} \delta p, \tag{9}$$

or

$$\frac{\partial}{\partial t}[x_p(t)] = G(t)x_p(t) + S(t), \text{denoting } x_p(t) = \frac{\partial x}{\partial p}(t), \quad \text{with } G(t) \in \mathcal{R}^{n\times n} \text{and } S(t), x_p(t) \in \mathcal{R}^{n\times m}, \tag{10}$$

with initial condition $x_p(t_0) = \frac{\partial x_0}{\partial p}$. (10) may be solved as an initial value problem to obtain the sensitivity matrix of $x(t)$ with respect to p as a function of time. Observe that (10) is a linear time-varying matrix differential equation driven by input $S(t)$. Note that if i^{th} columns of $x_p(t)$, $S(t)$ are denoted by $x_{p,i}(t)$ and $S_i(t)$, for $i = 1, \cdots, m$, then (10) can be written as the set of vector differential equations

$$\frac{\partial}{\partial t}[x_{p,i}(t)] = G(t)x_{p,i}(t) + S_i(t), \quad i = 1, \cdots, m, \tag{11}$$

with initial conditions $x_{p,i}(0)$.

If we are interested in the sensitivity of a weighted sum of the outputs with respect to p, *i.e.*, of $c^T(t)x(t)$, it is beneficial to use an adjoint formulation. First rewriting (11) as

$$\frac{\partial z}{\partial t} = G(t)z(t) + b(t), \tag{12}$$

consider its homogeneous part

$$\frac{\partial z}{\partial t} = G(t)z(t), \tag{13}$$

and let its state-transition matrix be denoted by $\Phi(t,t_0)$. Then it can be shown (*e.g.*, [12]) that the *adjoint* system

$$\frac{\partial w}{\partial t} = -G^T(t)w(t) \tag{14}$$

has a state-transition matrix $\Phi_a(t,t_0)$, given by

$$\Phi_a(t,t_0) = \Phi^T(t_0,t). \tag{15}$$

This property becomes useful when we consider that the solution to (12), with initial condition $z(t_0) = z_0$, can be expressed using $\Phi(t,t_0)$ as

$$z(t) = \Phi(t,t_0)z_0 + \int_{t_0}^{t} \Phi(t,\tau)b(\tau)\,d\tau, \tag{16}$$

hence $y(t) = c^T(t)z(t)$ can be written as

$$\begin{aligned} y(t) &= c^T(t)\Phi(t,t_0)z_0 + \int_{t_0}^{t} c^T(t)\Phi(t,\tau)b(\tau)\,d\tau \\ &= \left[\Phi^T(t,t_0)c(t)\right]^T z_0 + \int_{t_0}^{t} [\Phi^T(t,\tau)c(t)]^T b(\tau)\,d\tau \\ &= [\Phi_a(t_0,t)c(t)]^T z_0 + \int_{t_0}^{t} [\Phi_a(\tau,t)c(t)]^T b(\tau)\,d\tau \\ &= r^T(t_0,t)z_0 + \int_{t_0}^{t} r^T(\tau,t)b(\tau)\,d\tau, \end{aligned} \tag{17}$$

where $r(t,t_0)$ is the solution of (14) with initial condition $r(t_0) = c(t_0)$. Fixing a specific t_0 in (17) and also a specific $t > t_0$, we observe that the solution of (14) for $r(t_0,t)$ and $r(\tau,t)$ involves applying the initial condition at t, and solving *backward in time* upto t_0 or τ.

We note that evaluating (17) is in general relatively expensive, since it involves the solution of (14) over the interval $[t_0,t]$ for *each* value of t. For time-invariant systems, however (*i.e.*, $G(t) \equiv G$, and $\Phi(t,t_0) \equiv \Phi(t-t_0)$), the expense reduces enormously, as a *single* solution of (14) over the interval $[-(T-t_0),0]$ suffices to evaluate (17) over *all* $t \in [t_0,T]$.

3. RF SENSITIVITY CALCULATIONS

3.1. Harmonic Balance

Harmonic balance (HB) [13–16] is a method based on Fourier series expansions for solving periodic steady-state problems, and is widely employed for systems that are weakly or moderately nonlinear. More specifically, given (8) with periodic $b(t)$, *i.e.*, $b(t+T) = b(t)$, HB seeks to find a periodic solution $x(t)$ of (8) with the same period T. To this end, $b(t)$ and $x(t)$ are expanded in Fourier series, to $N = 2M+1$ terms:

$$\begin{aligned} b(t) &= \sum_{i=-M}^{M} B_i e^{ji\frac{2\pi}{T}t}, \\ x(t) &= \sum_{i=-M}^{M} X_i e^{ji\frac{2\pi}{T}t}. \end{aligned} \tag{18}$$

The Fourier coefficients $B_i, X_i \in \mathcal{R}^n$ can be conveniently collected in long vectors of size $n \times N$:

$$B = [B_{-M}, \cdots, B_M]^T, \quad X = [X_{-M}, \cdots, X_M]^T. \tag{19}$$

We now note that the term $f(x(t),p)$ in (8) is T-periodic if $x(t)$ is T-periodic, hence $f(x(t),p)$ can also be expanded in a Fourier series similar to (18), and the Fourier coefficients stacked up in a vector similar to (19). Denote this vector, *i.e.*, the vector of Fourier coefficients of $f(x(t),p)$ by $F(X,p)$.

Finally, we note that the term $\dot{x}$ in (8) is also T-periodic if $x(t)$ is T-periodic, and that its i^{th} Fourier coefficient is given by $ji\frac{2\pi}{T}X_i$. Hence the vector of Fourier coefficients of $\dot{x}$ is given by ΩX, where $\Omega = j\frac{2\pi}{T}\text{diag}(-MI_n,\cdots,MI_n)$, a diagonal matrix of size $n \times N$, with I_n the identity matrix of size n.

Harmonic balance consists simply of equating the Fourier coefficients (or harmonics) of the terms in (8), *i.e.*,

$$G(X,p) \equiv \Omega X - F(X,p) - B = 0. \qquad (20)$$

(20) is a nonlinear equation in $X \in \mathcal{R}^{n\times N}$, hence the sensitivity techniques in Section 2.1 apply. Specifically, if we are interested in the sensitivity of $y = c^T X$, we have

$$\delta y = s^T \delta p, \text{ with } s = -\left.\frac{\partial G}{\partial p}\right|_{X,p} r, \text{ and}$$
$$\overset{\text{HB}*}{\mathbb{J}}\, r = c, \text{ where } \overset{\text{HB}}{\mathbb{J}} = \left.\frac{\partial G}{\partial X}\right|_{X,p}. \qquad (21)$$

In view of (20), we note that $\overset{\text{HB}}{\mathbb{J}}$, the *HB Jacobian matrix*, has the form

$$\overset{\text{HB}}{\mathbb{J}} = \Omega - \left.\frac{\partial F}{\partial X}\right|_{X,p}. \qquad (22)$$

This matrix is typically formed and used in the process of solving (20) numerically for a nonlinear solution by, *e.g.*, a Newton-Raphson procedure. Hence the adjoint solution in (21), constituting the main computational step in finding the sensitivity δy, can readily be carried out using the available HB Jacobian matrix. The adjoint solution is especially convenient in the context of 'fast' HB techniques (*e.g.*, [9,17]), using which linear solutions of the type $\overset{\text{HB}}{\mathbb{J}}z = w$ are carried out efficiently with $O(nN\log N)$ computation. In particular, when the Lanczos-process-based QMR iterative linear solution method [18] is employed during fast solution, the adjoint system (21) is solved repeatedly during the nonlinear solution of (20), hence the essence of sensitivity computation for HB reduces to single extra call of an existing subroutine.

3.2. Shooting

Shooting [19–21] is another technique for solving for the periodic steady state of (8) when $b(t)$ is periodic, and is often preferred over HB when the system has strong nonlinearities or switching behaviour. Instead of using Fourier expansions, shooting employs a nonlinear fixed-point formulation to express periodicity. Assuming that (8) has a unique solution for every initial condition $x(t_0) = x_0$, we can write its solution in terms of its state-transition function Φ as

$$x(t) = \Phi(t;x_0,t_0,p). \qquad (23)$$

If (8) admits of a T-periodic solution under T-periodic excitation $b(t)$, then $x(t_0+T) = x(t_0)$; using (23), we obtain

$$x(t_0+T) = x(t_0) = x_0 = \Phi(t_0+T;x_0,t_0,p), \text{ or}$$
$$G(x_0,p) \equiv x_0 - \Phi(T;x_0,0,p) = 0, \quad \text{setting } t_0 = 0. \qquad (24)$$

(24) is a nonlinear equation in $x_0 \in \mathcal{R}^n$, hence the sensitivity techniques of Section 2.1 again apply. Specifically, to calculate the sensitivity of $y = c^T x_0$, we have

$$\delta y = s^T \delta p, \text{ with } s = -\left.\frac{\partial G}{\partial p}\right|_{x_0,p} r, \text{ and}$$
$$\overset{\text{TD}}{\mathbb{J}}{}^T r = c, \text{ where } \overset{\text{TD}}{\mathbb{J}} = \left.\frac{\partial G}{\partial x_0}\right|_{x_0,p}. \qquad (25)$$

Using (24), we note that the *shooting Jacobian matrix* $\overset{\text{TD}}{\mathbb{J}}$ is given by

$$\overset{\text{TD}}{\mathbb{J}} = I_n - \left.\frac{\partial \Phi}{\partial x_0}\right|_{T;x_0,0,p}. \qquad (26)$$

This matrix is formed and used in the process of solving (24) for a nonlinear solution. It can be shown that applications of $\overset{\text{TD}}{\mathbb{J}}$ to a vector r can be realized by solving (13) with initial condition $z(0) = r$ over the period $[0,T]$, *i.e.*, a single transient simulation of the linearization of (8). Likewise, application of $\overset{\text{TD}}{\mathbb{J}}{}^T$ to a vector r is the solution of (14) with 'initial condition' $w(T) = r$, backward in time over $[0,T]$ – also a single transient simulation of the linearized adjoint system. These observations are particularly useful when iterative linear algebra techniques are used in 'fast' methods for solving (24) [11], for (25) can then be computed efficiently using a small number of backward adjoint transient simulations and a single forward transient simulation of the linearized system. Once again, if Lanczos-based iterative linear solution techniques such as QMR are already employed in the solution of the nonlinear shooting equation (24), subroutines for these forward and backward transient simulations are already available.

3.3. Envelope simulation

An important task in RF simulation is to go beyond finding a periodic steady state of (8) to a broader class of responses, featuring slowly-changing 'envelopes' riding on fast-varying periodic waveforms. An example is $b(t) = e^{-t}\sin(2\pi 10^4 t)$, *i.e.*, a 10kHz sinusoid damped by a slowly-decaying exponential; e^{-t} is the slow envelope in this case. In myriad applications, especially from communications, aperiodic information signals are often carried in such envelopes, and it is of interest to find the sensitivities of these envelopes as they propagate through a system.

While various envelope calculation techniques are available [22–25], for brevity we shall focus in this section on *Fourier envelope* techniques, *i.e.*, where the fast 'carrier' variations are expanded in Fourier series. We stress that the procedure outlined here carries over to other bases, *e.g.*, locally supported time-domain bases. For convenience, we shall use a multi-time formulation (*e.g.*, [26,27]) to develop the sensitivity procedure for envelope computations.

A two-time MPDE [27] corresponding to (8) is

$$\left[\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right]\hat{x}(t_1,t_2) = f(\hat{x},p) + \hat{b}(t_1,t_2). \qquad (27)$$

We express the assumption that the excitation $\hat{b}$ and the response $\hat{x}$ are in envelope form by requiring that both are

T-periodic in the t_2 argument, with the variation in t_1 being aperiodic and much slower. Hence we can express the t_2 variation in Fourier series as in (18), collecting the Fourier coefficients (now functions of t_1) in vectors $B(t_1)$ and $X(t_1)$, respectively, as in (19). Following the same procedure used to obtain (20) concentrating only on the t_2 variations, the *Fourier envelope equation* for (8) is obtained to be

$$\begin{aligned}\frac{\partial X(t_1)}{\partial t_1} + \Omega X(t_1) &= F(X(t_1),p) + B(t_1), \text{ or}\\ \frac{\partial X(t_1)}{\partial t_1} &= (F(X(t_1),p) - \Omega X(t_1)) + B(t_1).\end{aligned} \tag{28}$$

Ω and $F(\cdot,\cdot)$ in (28) are the same as in (20); $B(t_1)$ are the envelopes of the inputs to the system. Note that (28) is in the same form as (8), hence the adjoint sensitivity procedure of Section 2.2 applies.

In particular, computing the sensitivity of $y(t_1) = c^T(t_1)X(t_1)$ requires the solution of the linearized adjoint equation corresponding to (28) for each t_1, a computationally intensive process. The procedure simplifies considerably, however, when sensitivities about a steady envelope solution are desired, *i.e.*, when $B(t_1)$ and $X(t_1)$ are constant, independent of t_1. In this case, the linearization of (28) becomes time-invariant, and the transient sensitivity can be found using a single backward solution of the linearized adjoint equation.

4. CONCLUSION

We have used the theory of adjoint linear systems to obtain sensitivity calculation techniques for common RF analyses. We have shown that the benefits of recent 'fast' methods for these analyses, employing iterative linear solution at their core, carry over to finding sensitivities, particularly when they are already based on adjoints.

5. REFERENCES

[1] S.W. Director and R.A Rohrer, "The Generalized Adjoint Network and Network Sensitivities," *IEEE Trans. Ckt. Theory*, vol. 16, pp. 318–323, Aug. 1969.

[2] D.A. Hocevar, P. Yang, T.N. Trick, and B.D. Epler, "Transient Sensitivity Computation for MOSFET Circuits," *IEEE Trans. CAD*, vol. 4, pp. 609–620, Oct. 1985.

[3] U. Choudhury, "Sensitivity Computation in SPICE3," M.S. thesis, EECS Dept., Univ. Calif. Berkeley, Elec. Res. Lab., Dec. 1988.

[4] J.W. Bandler, Q.J. Zhang, and R.M. Biernacki, "A unified theory for frequency-domain simulation and sensitivity analysis of linear and nonlinear circuits," *IEEE Trans. MTT*, vol. 36, no. 12, pp. 1661–1669, Dec 1988.

[5] E. Ngoya, M. Gayral, and J. Rousset, "Application of spectral-balance to the design of non-linear microwave circuits. Steady-state response, sensitivity analysis and optimization," in *Proc. IEEE ISCAS*, 1988, vol. 3, pp. 2283–2285.

[6] J.W. Bandler, Q.J. Zhang, and R.M. Biernacki, "Practical high speed gradient computation for harmonic balance simulators," in *Proc. IEEE MTT Symp.*, 1989, vol. 1, pp. 363–366.

[7] S.W Ng, S.C. Wong, and Y.S. Lee, "Small signal simulation of switching converters," *IEEE Trans. Ckts. Syst. – I: Fund. Th. Appl.*, vol. 46, pp. 731–739, June 1999.

[8] T. Kato and W. Tachibana, "Periodic steady-state analysis by a modified shooting method for a power electronic circuit," in *Proc. Power Conversion Conference – Nagaoka*, 1997, vol. 1, pp. 367–372.

[9] Markus Rösch, *Schnell Simulation des stationären Verhaltens nichtlinearer Schaltungen*, Ph.D. thesis, Technischen Universität München, 1992.

[10] D. Long, R.C. Melville, et al, "Full chip harmonic balance," in *Proc. IEEE CICC*, May 1997.

[11] R. Telichevesky, K. Kundert, and J. White, "Efficient steady-state analysis based on matrix-free krylov subspace methods," in *Proc. IEEE DAC*, 1995, pp. 480–484.

[12] R.A. DeCarlo, *Linear Systems*, Prentice Hall, NJ, 1989.

[13] L.O. Chua and A. Ushida, "Algorithms for computing almost periodic steady-State response of nonlinear systems to multiple input frequencies," *IEEE Trans. Ckts. Syst.*, vol. CAS-28, no. 10, pp. 953–971, Oct 1981.

[14] Kenneth S. Kundert and Alberto Sangiovanni-Vincentelli, "Simulation of nonlinear circuits in the frequency domain," *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, vol. CAD-5, no. 4, pp. 521–535, Oct. 1986.

[15] K.S. Kundert, J.K. White, and A. Sangiovanni-Vincentelli, *Steady-state methods for simulating analog and microwave circuits*, Kluwer Academic Publishers, 1990.

[16] R.J. Gilmore and M.B. Steer, "Nonlinear circuit analysis using the method of harmonic balance – a review of the art. Part I. Introductory concepts.," *Int. J. on Microwave and Millimeter Wave CAE*, vol. 1, no. 1, 1991.

[17] R.C. Melville, P. Feldmann, and J. Roychowdhury, "Efficient multi-tone distortion analysis of analog integrated circuits," in *Proc. IEEE CICC*, May 1995, pp. 241–244.

[18] R. W. Freund and N. M. Nachtigal, "QMR: a quasi-minimal residual method for non-Hermitian linear systems," *Numerische Mathematik*, vol. 60, pp. 315–339, 1991.

[19] T.J. Aprille and T.N. Trick, "Steady-state analysis of nonlinear circuits with periodic inputs," *Proc. IEEE*, vol. 60, no. 1, pp. 108–114, Jan. 1972.

[20] S. Skelboe, "Computation of the periodic steady-state response of nonlinear networks by extrapolation methods," *IEEE Trans. Ckts. Syst.*, vol. CAS-27, no. 3, pp. 161–175, Mar. 1980.

[21] K.S. Kundert, *The Designer's Guide to Spice and Spectre*, Kluwer Academic Publishers, 1995.

[22] L. Petzold, "An efficient numerical method for highly oscillatory ordinary differential equations," *SIAM J. Numer. Anal.*, vol. 18, no. 3, June 1981.

[23] D. Sharrit, "New Method of Analysis of Communication Systems," *MTTS WMFA: Nonlinear CAD Workshop*, June 1996.

[24] P. Feldmann and J. Roychowdhury, "Computation of circuit waveform envelopes using an efficient, matrix-decomposed harmonic balance algorithm," in *Proc. ICCAD*, Nov. 1996.

[25] K. Kundert, J. White, and A. Sangiovanni-Vincentelli, "An envelope-following method for the efficient transient simulation of switching power and filter circuits," in *Proc. ICCAD*, Nov. 1988.

[26] H.G. Brachtendorf, G. Welsch, R. Laur, and A. Bunse-Gerstner, "Numerical steady state analysis of electronic circuits driven by multi-tone signals," *Electrical Engineering (Springer-Verlag)*, vol. 79, pp. 103–112, 1996.

[27] J. Roychowdhury, "Analysing circuits with widely-separated time scales using numerical PDE methods," *IEEE Trans. Ckts. Syst. – I: Fund. Th. Appl.*, May 2001.