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2C-3 Analytical Expressions for Phase Noise Eigenfunctions of LC Oscillators
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Analytical Expressions for Phase Noise Eigenfunctions of LC Oscillators

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Abstract

We obtain analytical expressions for eigenfunctions that characterize the phase noise performance of generic LC oscillator structures. Using these, we also obtain analytical expressions for the timing jitter and spectrum of such oscillators. Our approach is based on identifying *three fundamental parameters*, derived from the oscillator's steady state, that characterize these eigenfunctions. Our analysis accounts for the nonlinear mechanism that stabilizes oscillator amplitudes. It also lays out, quantitatively and in analytical form, how *symmetry* in an LC oscillator's negative resistance mechanism impacts the oscillator's eigenfunctions and its phase noise/jitter characteristics. We show that symmetry results in particularly simple forms for the PPV and resultant phase noise. We compare our expressions with existing LC oscillator design formulae and show that the expressions match for symmetric nonlinearities. We validate our analytical results against simulation on practical CMOS LC oscillator circuits. Our expressions and symmetry results are expected to be useful tools for optimizing phase noise performance during the design of LC oscillators.

1 Introduction

Oscillators are systems that, spontaneously and without external input, generate a periodic signal indefinitely. They are omnipresent in electronic circuits; for example, they are an integral part of phase-locked loops, clock recovery circuits, frequency synthesizers and so on. Several classes of electronic oscillators exist (*e.g.*, ring, relaxation, LC, *etc.*); of these, LC oscillators are arguably the most important for high-performance and communication applications.

An important concern during oscillator design is to quantify its noise performance, particularly its *phase noise* and *jitter* [2, 8, 10, 11]. Phase noise in oscillators significantly affects its time-domain and frequency-domain properties. In the time domain, it results in uncertainties in switching or timing instants. This phenomenon, called jitter, is of serious concern, especially in digital systems synchronized to clock signals from oscillators. The same effect, viewed in the frequency domain, results in the spread of discrete tones to a continuous range of neighboring frequencies. This spread, commonly termed phase noise, is a serious problem in communication circuits. Since the design of low-noise LC oscillators is a topic of current interest in the analog/mixed-signal design community, characterizing phase noise and jitter *correctly* is of crucial practical importance.

There has been a great deal of interest in the phase noise problem over decades (see [2] for a review). The most recent and rigorous theory that addresses the phenomenon quantitatively has resulted in efficient numerical techniques for simulating phase noise at the SPICE level [2]. However, during early design and analysis of oscillators, *analytical formulae* for phase noise are of great value. Previous analytical formulae for LC oscillators (see Section 2 for a brief review) rely on approximations that do not fully account for the fundamental nonlinear mechanism of an oscillator's dynamics. Such formulae, while valuable for certain LC topologies and parameter choices, lose their predictive ability for others. In particular, some of the previous approaches do not distinguish correctly between *symmetric* and *asymmetric* nonlinearities, resulting in inaccurate formulae for the latter case.

In this paper, we develop more general and accurate analytical expressions for phase noise and jitter in LC oscillators than previously available. Our approach is based on starting from the rigorous nonlinear theory of [2] and following a step-by-step mathematical procedure with only a single, clearly identified, approximation. We show that this approximation (truncation of a Fourier series) is valid to a high degree for practical LC oscillators with reasonable Q factors. For oscillators with arbitrary nonlinear *i*-*v* characteristics, we show that the *DC component and first two harmonics of the incremental conductance of the oscillator nonlinearity*, when excited sinusoidally, are parameters critical in determining phase noise and jitter characteristics. First, we obtain analytical expressions for phase noise eigenfunctions, *i.e.*, the Perturbation Projection Vector (PPV) [2], in terms of these three fundamental parameters (which can be obtained numerically or analytically for a given oscillator). Using these analytical PPV forms, we then obtain formulae for jitter and noise spectrum.

Further, we investigate the special case of oscillators with symmetric nonlinear *i*-*v* characteristics, and show that for such oscillators, the three harmonic parameters above drop out, resulting in simple formulae that are identical to those obtained by Hajimiri et al [10]. We emphasize that these simple formulae do not rely on circuit structural symmetry, only on symmetry in the nonlinear *i*-*v* characteristic, which have no relation to each other. We reiterate that when the nonlinearity is not symmetric, the simple expressions are invalid; our more general formulae, involving the three parameters mentioned above, must be employed. We present results that verify the accuracy of our analytical formulae by comparing against numerical simulations carried out for practical LC oscillator circuits.

The remainder of the paper is organized as follows. In Section 2, we briefly review previous work on analytical models for phase noise in oscillators. In Section 3 we discuss the nonlinear operating mechanism of a generic LC oscillator, thus providing insight and motivating an approximation employed later. Next, in Section 4, we form the steady-state Jacobian matrix of the oscillator analytically, prove its singularity, and use this fact to obtain analytical expressions for the PPV. In Section 5, we use the PPV expressions to obtain analytical formulae for oscillator's jitter and power spectrum and, for the symmetric case, compare them against earlier analytical models [11, 14]. Finally, in Section 6, we present results to verify the accuracy of our analytical formulae against the numerical simulations for a CMOS LC oscillator and a tanh-based negative resistance oscillator.

2 Previous Work

An established body of literature, developed over decades, is available on various aspects of the phase noise problem [2,3,10,11,14]. Here, we focus on previous work on developing analytical formulae for phase noise in terms of noise source values and circuit parameters. Though relevant to our discussion, we do not provide a discussion of the studies contributing to the theory as they are not the central to our contribution, but we refer the interested reader to [2, 12].

Leeson [11, 12] was first to propose a simple but extremely useful phenomenological model for phase noise, in terms of the noise currents of the oscillator's circuit elements and empirical parameters. He noted that the power spectrum of an oscillator consists

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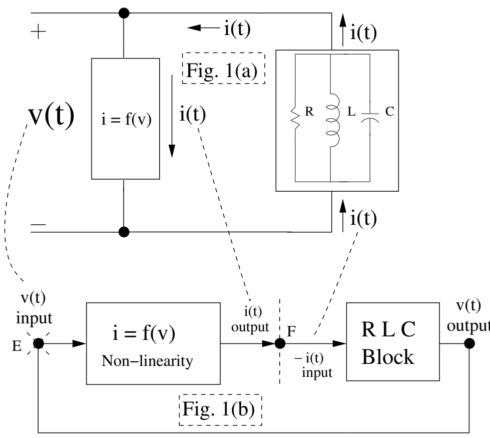

Figure 1: Generic LC oscillator circuit

of a $\frac{1}{f^2}$ region, which he identified as being caused by the white noise sources in the circuit. He also noted that, close to the oscillation frequency, flicker noise sources in the oscillator affected its spectrum. Cutler [13] confirmed that flicker ($\frac{1}{f}$) noise results in a $\frac{1}{f^3}$ region that dominates the $\frac{1}{f^2}$ shape close to the oscillation frequency. The Leeson and Leeson-Cutler models are empirical and have limitations; for example, they fail to predict phase noise power correctly at frequencies close to the carrier frequency.

Craninckx and Steyaert [14] arrived at phase noise expressions by employing linear time-invariant (LTI) analysis. They identified the noise contributions of the individual components of the LC tank and arrived at expressions largely similar to Leeson's expressions. Razavi [8] retained the LTI analysis but in addition he considered different categories of noise sources (such as additive noise sources, high frequency multiplicative noise sources and low frequency multiplicative noise sources) to develop more accurate models for phase noise in relaxation oscillators. The LTI models discussed above fail to account properly for the amplitude limiting mechanism inherent to oscillators, in addition to having the same drawback as Leeson-Cutler near and at the oscillator's center frequency.

Hajimiri's [10] phase noise model is one of the more recent and accurate analytical models for phase noise in LC oscillators. Based on a linear time-varying (LTV) analysis, Hajimiri obtained analytical formulae for phase noise in terms of noise currents and Fourier coefficients of a function called the impulse sensitivity function (ISF).

Thus while oscillators are nonlinear time-varying systems, the majority of the available analytical phase noise models are based on either *linear time-invariant* or *linear time-varying* theories.

A recent rigorous *nonlinear* theory for phase noise [2], applicable to oscillators of all types, has been successful in addressing fundamental deficiencies in linear phase noise analysis approaches. However, little has been done to translate this theory to obtain analytical formulae for phase noise. In this paper, we address the task of applying this theory to obtain analytical expressions for phase noise in LC oscillator structures.

3 Generic Feedback LC Oscillator

A generic LC oscillator structure is shown in Fig. 1(a). The $i = f(v)$ nonlinearity [4, 5] represents the active part of the oscillator that typically comprises of the MOS or BJT devices in parallel with the passive *RLC* tank circuit. The operation of the generic LC oscillator can be explained from the standpoint of the negative resistance concept. The $i = f(v)$ nonlinearity in effect presents a negative resistance to the circuit. Negative resistance signifies that as the applied voltage v increases the current i drawn by the circuit decreases (Fig. 2). In the absence of *i*-*v* nonlinearity, the tank circuit when excited by a noise impulse responds with a decaying oscillatory behavior due to heat losses in the resistive part of the tank. In the presence of noise impulses, if the *i*-*v* nonlinearity presents a negative resistance that matches (and cancels) the equivalent parallel resistance of the tank circuit, the tank oscillates indefinitely. Under such conditions it can be shown that Barkhausen criteria [7] that are the necessary conditions for a feedback circuit to oscillate are satisfied [16]. The Barkhausen criteria are given by: a) The magnitude of the loop gain around the closed loop of a feedback circuit should be equal to 1 and b) The total phase shift for a signal around the closed loop of a feedback circuit should be equal to 2π.

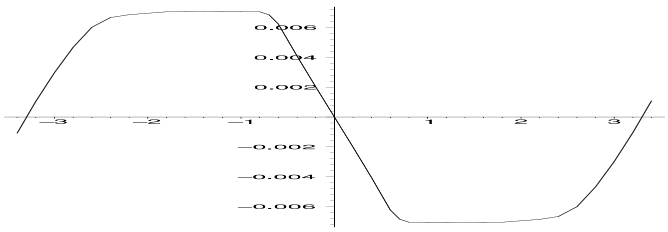

Figure 2: $i\,(vertical) - v\,(horizontal)$ nonlinearity plot from Spice

For the purpose of analyzing the oscillator's phase noise, let's consider Fig. 1(b), which is simply an equivalent representation of the mechanism by which the LC oscillator operates. The structure in Fig. 1(b) is a self-exciting feedback circuit with the feedback provided at node E. The $i = f(v)$ part of the circuit acts as a nonlinear time-varying process that takes in voltage $v(t)$ as an input and generates current $i(t)$ at its output. The RLC tank circuit forms a LTI block which takes in current $-i(t)$ as input and generates voltage $v(t)$ at its output. A mirror node F exists between the two blocks in Fig. 1(b), where current undergoes a π phase change. The reason the node F exists is due to the direction in which the current $i(t)$ enters the RLC tank in Fig. 1(a), as against the polarity of the voltage v across the tank circuit. The π phase shift due to F is necessary for the oscillator so as to have a total phase shift of 2π around the closed loop as per the Barkhausen criterion.

For the generic LC oscillator structure of Fig.1, we assume a steady state voltage of $v(t) = A\ \ cos(\omega_0 t)$, where A is a analytical variable which is a function of the circuit parameters. This assumption of considering just the first harmonic of the steady state voltage is well justified as the RLC tank circuit filters out the higher-order harmonics in the steady state voltage. The dc component of the steady state solution is taken as zero, because the inductor current would build up indefinitely otherwise.

Let's consider some special Lemma's that are relevant to this section and are applicable to any LC oscillator with an odd-symmetric $i = f(v)$ function. Lemmas 3.1 and 3.2 together show that a phase shift of π exists from $v(t)$ to $i(t)$ if the function $i = f(v)$ is an odd-symmetric function. Lemma 3.3 gives the widely-known formula for the fundamental oscillation frequency for *i*-*v* symmetric LC oscillators. The proofs for the Lemma's are omitted for lack of space.

Lemma 3.1 *The phase shift from $v(t)$ to $i(t)$ for the function $i = f(v) = k\ v$, is either 0 or π depending on whether k is a positive or negative constant*

Lemma 3.2 *For the generic LC oscillator circuit, there is a phase shift of π from $v(t)$ to $i(t)$ if $i = f(v)$ is an odd-symmetric function.*

Lemma 3.3 *For the generic LC oscillator circuit, the oscillation frequency is given by the resonance frequency of the tank circuit $\omega_0 = \frac{1}{\sqrt{LC}}$, if $i = f(v)$ is an odd-symmetric function*

Note: Lemmas 3.2 and 3.3 reinforce the fact that a node F exists in Fig. 1(b) at which there is a phase shift of π. From the Lemmas, we can see that for any LC oscillator (with just the first harmonic in the steady state voltage) the *i*-*v* block presents a π phase shift, and the *RLC* block a zero phase shift as $\omega_0 L = \frac{1}{\omega_0 C}$. Since the phase shift around the loop of a oscillator should be 2π according to the Barkhausen criterion, it is obvious that a node F which provides a π phase shift exists such that the circuit oscillates.

4 Obtaining the PPV of the Oscillator from the Steady State Analytical Solution

Once we have the analytical steady state solution of the oscillator, the next step in our procedure is to determine the phase noise eigenfunctions or the Perturbation Projection Vector (PPV) [2] of the generic LC oscillator. For obtaining the PPV, the Harmonic Jacobian matrix J_{HB} is first formed using the steady state analytical solution of the oscillator. We then prove that the J_{HB} matrix is singular, and use the condition for singularity to solve for the null-space of the adjoint of J_{HB}, which gives us the fourier coefficients of the PPV.

The Lemmas below lead us to finding the PPV of the generic oscillator. Lemma 4.1 gives the Harmonic Balance equations for the generic LC oscillator. We consider only the dc term and the first positive and the negative harmonic components for the state variables $v(t)$ (voltage across the capacitor) and $i_L(t)$ (current through the inductor), and the current $i(t) = f(v(t))$. Considering just the dc term and first positive and the negative harmonic components suffices, as the effect of the higher order harmonics on calculating the PPV and the PPV itself is negligible, as we will see from the numerical simulations in Section 6.

Lemma 4.1 *The system of state equations of the generic oscillator circuit of Fig. 1 are given by*

$$\begin{aligned} \frac{V_0}{R} + I_{L0} + Y_0(\overline{V_1}, V_0, V_1) &= 0 \\ V_1\left(\frac{1}{R} + j\omega_0 C\right) + I_{L1} + Y_1(\overline{V_1}, V_0, V_1) &= 0 \\ \overline{V_1}\left(\frac{1}{R} - j\omega_0 C\right) + \overline{I_{L1}} + \overline{Y_1}(\overline{V_1}, V_0, V_1) &= 0 \end{aligned} \tag{1}$$
$$V_0 = 0, \qquad I_{L1}(j\omega_0 L) - V_1 = 0, \qquad -\overline{I_{L1}}(j\omega_0 L) - \overline{V_1} = 0$$

V_0, V_1, $\overline{V_1}$ *are respectively the dc term, first fourier coefficient for positive frequencies and the first fourier coefficient for negative frequencies of* $v(t)$*; and* Y_1, Y_1, $\overline{Y_1}$ *are respectively the dc term, first fourier coefficient for positive frequencies and the first fourier coefficient for negative frequencies of* $i(t) = f(v(t))$*. The above equations can be arranged in the form of*

$$H(X) = 0 \tag{2}$$

where $X^T = [V_0 \;\; V_1 \;\; \overline{V_1} \;\; I_{L0} \;\; I_{L1} \;\; \overline{I_{L1}}]$

Lemma 4.2 shows that a phase shift in the state variables simply translates to the exact same phase shift in $i(t) = f(v(t))$. This result leads to the proof that the Harmonic Jacobian matrix J_{HB} is singular as discussed further in this Section.

Lemma 4.2

$$H(D_\theta X) = D_\theta\, H(X) \tag{3}$$

where $D_\theta = diag(1, e^{j\theta}, e^{-j\theta}, 1, e^{j\theta}, e^{-j\theta})$

Note: The Harmonic Jacobian matrix of the generic oscillator is given by ($a = \frac{1}{R} + \gamma_0 + j\omega_0 C$)

$$J_{HB} = \frac{\partial H}{\partial(V_0, V_1, \overline{V_1}, I_{L0}, I_{L1}, \overline{I_{L1}})} = \begin{pmatrix} \frac{1}{R}+\gamma_0 & \gamma_1 & \overline{\gamma_1} & 1 & 0 & 0 \\ \overline{\gamma_1} & a & \overline{\gamma_2} & 0 & -1 & 0 \\ \gamma_1 & \gamma_2 & \overline{a} & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & j\omega_0 L & 0 \\ 0 & 0 & 1 & 0 & 0 & -j\omega_0 L \end{pmatrix} \tag{4}$$

where γ_0, γ_1, γ_2 are the dc term and the first two harmonics of the $R_r(t)$, which is defined as the incremental conductance of the oscillator nonlinearity valued at the steady state voltage $v(t) = A\,cos(\omega_0 t)$.

$$\begin{aligned} R_r(t) &= \frac{\partial f(v(t))}{\partial v}\Big|_{v(t)} = \frac{\partial y(t)}{\partial V_0}, \;\; \gamma_0 = \frac{1}{T}\int_0^T R_r(t)\,dt \\ \gamma_1 &= Re(\gamma_1) + jIm(\gamma_1) = \frac{1}{T}\int_0^T R_r(t)\,e^{j\omega_0 t}\,dt \\ \gamma_2 &= Re(\gamma_2) + jIm(\gamma_2) = \frac{1}{T}\int_0^T R_r(t)\,e^{j2\omega_0 t}\,dt \end{aligned} \tag{5}$$

Lemma 4.3 shows that the matrix J_{HB} and its adjoint $J_{HB}{}^*$ are singular. From Lemma 4.2, we can clearly see that a continuum of phase shifts in the state variables $v(t)$ and $i_L(t)$ leads to a continuum of phase shifts in $i(t) = f(v(t))$. Hence, the derivative of the H(X) with respect to the phase shift is zero. Lemma 4.3 follows from this argument.

Lemma 4.3 J_{HB} *(and hence* $J_{HB}{}^*$*) are singular.*

Lemma 4.4 gives the Perturbation Projection Vector (PPV) of the generic oscillator. Solving for the null-space of the $J_{HB}{}^*$ matrix, *i.e.* solving the equation $J_{HB}{}^* \times P = 0$ gives us P, which is a column vector of the fourier coefficients of the PPV.

Lemma 4.4 *The PPV vector of the generic LC oscillator circuit is given by*

$$PPV(t) = \begin{pmatrix} p_2\,e^{-j\,\omega_0 t} \; + \; p_0 \; + \; p_1\,e^{j\,\omega_0 t} \\ p_5\,e^{-j\,\omega_0 t} \; + \; p_3 \; + \; p_4\,e^{j\,\omega_0 t} \end{pmatrix} \tag{6}$$

where p_0, p_1, p_2 *are respectively the dc term, first positive and the first negative fourier coefficients of the PPV of* $v(t)$ *and* p_3, p_4, p_5 *are respectively the dc term, first positive and the first negative fourier coefficients of the PPV of* $i_L(t)$ *defined as*

$$\begin{aligned} & p_0 = 0,\, p_1 = \overline{\beta},\, p_2 = \beta,\, p_3 = -\gamma_1\,\overline{\beta} - \overline{\gamma_1}\,\beta \\ & p_4 = -\frac{j\,\overline{\beta}}{\omega_0 L},\, p_5 = \frac{j\,\beta}{\omega_0 L} \quad \textit{and} \\ & \beta = K\left(\omega_0 C - \frac{1}{\omega_0 L} - Img(\gamma_2) + j\left(\frac{1}{R} + \gamma_0 + Re(\gamma_2)\right)\right) \end{aligned} \tag{7}$$

where K is a real constant determined by the orthonormality condition $PPV(t)^T \times \dot{x_s}(t) = 1$ *and* $x_s(t)$ *is the steady state vector.* $\dot{x_s}(t)$ *is given by* $\dot{x_s}(t)^T = [C\dot{v}(t) \;\;\; L\dot{i_L}(t)]$

Lemma 4.5 gives the PPV of LC oscillators with odd-symmetric *i-v* nonlinearities in terms of the circuit parameters. .

Lemma 4.5 *The PPV of any LC oscillator with an odd-symmetric i-v nonlinearity and with a steady state voltage* $v(t) = A\,cos(\omega_0\,t + \theta)$*, where* $0 \le \theta \le 2\Pi$ *is given by*

$$PPV(t) = \begin{pmatrix} -\sqrt{\frac{L}{C}}\,\frac{1}{A}\,sin(\omega_0\,t + \theta) \\ \frac{1}{A}\,cos(\omega_0\,t + \theta) \end{pmatrix} \tag{8}$$

where L is the inductance, C the capacitance, ω_0 *the fundamental oscillation frequency and A the amplitude of the steady state voltage.*

5 Obtaining the Phase Noise Jitter and Power Spectrum

The generic LC oscillator circuit with a noise source $b(t) = N\,n(t)$ is shown in Fig. 3 where $n(t)$ is a noise source of unit double-sided Power Spectral Density (PSD) and N is a constant that indicates the intensity of the noise source. The scalar constant c [2] that characterizes phase noise analytically is given by

$$c = \sum_{i=1}^{p} \frac{1}{T}\int_0^T \left[v_1{}^T(\tau)B_i(\tau)\right]^2 d\tau = \sum_{i=1}^{p} c_i \tag{9}$$

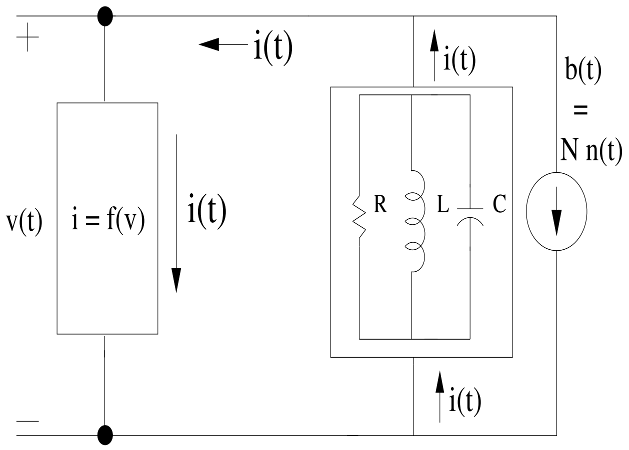

Figure 3: Generic Oscillator with noise

where $B(\cdot) : \mathbf{R}^n \rightarrow \mathbf{R}^{n \times p}$ represents the modulation of the intensities of the noise sources, p is then number of noise sources i.e., the column dimension of $B(x_s(\cdot))$, and $B_i(\cdot)$ is the i^{th} column of $B(x_s(\cdot))$ which maps the i^{th} noise source to the equations of the system.

Lemma 5.1 gives the expression for noise constant c in terms of the parameters γ_0, γ_1 and γ_2 and the circuit parameters.

Lemma 5.1 *The noise characterization constant c in the presence of noise source $N\,n(t)$ is given by*

$$c = 2N^2K^2\left((\omega_0 C - \frac{1}{\omega_0 L} - Img(\gamma_2))^2 + (\frac{1}{R} + \gamma_0 + Re(\gamma_2))^2\right) \tag{10}$$

where K is the real constant from Lemma 4.4 obtained from the orthonormality condition $PPV(t)^T \times \dot{x_s}(t) = 1$ and $x_s(t)$ is the steady state vector.

Lemma 5.2 gives the formula for c in LC oscillators with odd-symmetric *i-v* nonlinearities in terms of the circuit parameters.

Lemma 5.2 *The noise characterization constant c in LC oscillators with odd-symmetric i-v nonlinearities in the presence of a noise source $N\,n(t)$ is given by*

$$c = \frac{N^2\,\omega_0^2\,L^2}{2\,A^2} = \frac{N^2}{2\,\omega_0^2\,C^2\,A^2} = \frac{N^2}{2}\,\frac{L}{C}\,\frac{1}{A^2} \tag{11}$$

where L is the inductance, C the capacitance, ω_0 the fundamental oscillation frequency and A is the amplitude of the steady state voltage, $n(t)$ is a noise source of unit double-sided PSD and N is the intensity of the noise source.

NOTE: The PSD of the phase noise around the first harmonic is typically of interest to designers. The single-sideband phase noise power denoted by $\mathcal{L}(f_m)$ measured in dBc/Hz which is widely-used [2] is defined as

$$\begin{aligned} \mathcal{L}(\Delta\omega) &\approx 10 log_{10}\left(\frac{f_0{}^2\,c}{\pi^2 f_0{}^4\,c^2 + f_m{}^2}\right),\ 0 \le f_m \ll f_0 \\ &\approx 10 log_{10}\left(\left(\frac{f_0}{f_m}\right)^2 c\right),\ \pi f_0{}^2 c \ll f_m \ll f_0 \end{aligned} \tag{12}$$

where $\omega_0 = 2\pi f_0$ is the fundamental oscillation frequency and $\Delta\omega = 2\pi f_m$ is the offset from the fundamental oscillation frequency.

As a special case, let's consider the phase noise of *i-v* symmetric oscillators due to only the parallel resistance of the tank, in the vicinity of the first harmonic of the oscillation frequency. The double-sided PSD of the noise source due to the parallel resistance R of the tank circuit is given by $N^2 = \overline{i_{n1}{}^2}/f = \frac{2kT}{R}$. From (12) and using the result from Lemma 5.2, we get an expression for the phase noise power in this case as

$$\mathcal{L}(\Delta\omega) = 10\ log_{10}\left(\frac{kT}{A^2}\,\frac{1}{R(\omega_0 C)^2}\left(\frac{\omega_0}{\Delta\omega}\right)^2\right) \tag{13}$$

which matches with the phase noise expression derived by Hajimiri et al. [10] for the case of a generic LC oscillator in which noise is contributed only by the parallel resistance of the tank.

The double-sided PSD of the total noise for the generic LC oscillator (includes the noise contributions of the active devices and the parallel resistance) can be approximated by $N^2 = \overline{i_{n2}{}^2}/f = \frac{2FkT}{R}$, where F is called the device excess noise factor. For this case, we get the expression for the phase noise power as

$$\mathcal{L}(\Delta\omega) = 10\ log_{10}\left(\frac{2FkT}{P_{avg}}\left(\frac{\omega_0}{2\,Q_L\,\Delta\omega}\right)^2\right) \tag{14}$$

where $P_{avg} = A^2/(2R)$ is the average power consumed in the parallel resistance of the tank. This expression matches with the phase noise expression of Leeson-Cutler model [11] for the $\frac{1}{f^2}$ region of the frequency spectrum.

NOTE: The variance of the timing jitter [2] for the generic LC oscillator structure in the presence of phase noise is given by

$$E[(t_m - m\,T)^2] = c\,m\,T \tag{15}$$

where $t_m = m\,T$, $m = 1, 2...$ are the times of clock transitions in a perfect oscillator unaffected by phase noise and T is the clockperiod.

From Lemma 5.2 and (15), we can get a simple expression for the variance of timing jitter in LC oscillators with symmetric *i-v* nonlinearities as

$$E[(t_m - m\,T)^2] = \frac{N^2}{2}\,\frac{L}{C}\,\frac{1}{A^2}\,m\,T \tag{16}$$

In this Section, we can observe that the simple expressions derived for the phase noise power spectrum and jitter in (14), (13) and (16) are valid only for i-v symmetric oscillators. For the LC oscillators with asymmetric i-v nonlinearities, the analytical expression for c and hence the analytical expressions for the power spectrum and the timing jitter contain the factors γ_0, γ_1 and γ_2 (derived from the incremental conductance of the oscillator's nonlinearity) and the circuit parameters.

6 Validation of analytical expressions against numerical simulation

The analytical formulae derived in the previous sections are verified against numerical simulations [2] for two cases of practical LC oscillators described by the following state equations (the function $f(\cdot)$ differs for the two oscillators).

$$\begin{aligned} C\frac{dv}{dt} + f(v) + \frac{v}{R} + i_L &= 0 \\ L\frac{di_L}{dt} - v &= 0 \end{aligned} \tag{17}$$

The simulations for each oscillator were done for two cases:(a) Using the dc term and the first 30 positive and 30 negative harmonics for each of the state variables and the function $i(t) = f(v(t))$ (61 Harmonic components in all for each variable), and (b) Using the dc term and the first positive and negative harmonics for each of the state variables and the function $i(t)$ (3 Harmonic components in all for each variable).

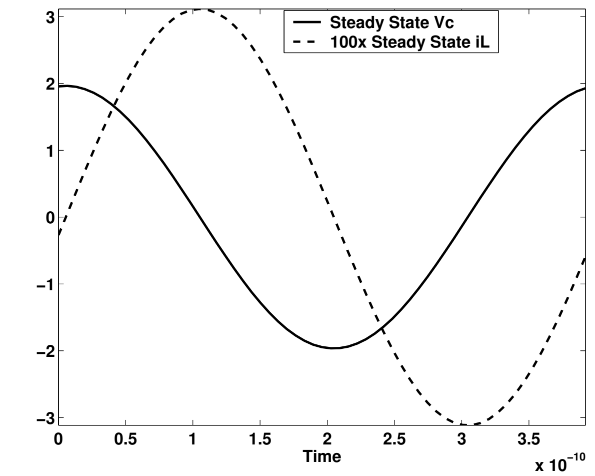

Figure 4: SS voltage $v(t)$, current $i_L(t)$ - CMOS oscillator

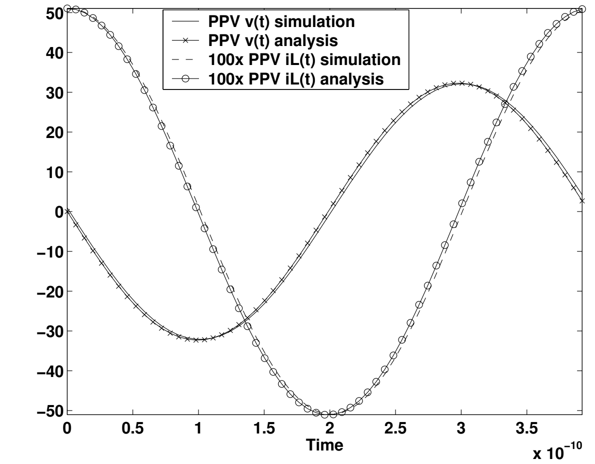

Figure 5: Simulation for PPV - 61 Components - CMOS oscillator

6.1 2.5 GHz, Cross-coupled CMOS LC Oscillator

The oscillator used in this case is a 2.5 GHz. cross-coupled CMOS LC oscillator circuit designed by Bunch et al. [4] with an odd-symmetric $i = f(v)$ function. The $i = f(v)$ nonlinearity for this LC oscillator was obtained by Bunch et al. [5] by fitting a least-squares curve to the $i - v$ curve obtained from SPICE simulation. The values of the circuit elements are L = 4 nH and C = 1 pF and the equivalent parallel resistance of the tank circuit R = 150 Ω. The steady state solution of the circuit (Fig.4) was obtained from a transient simulation as well as by Harmonic Balance as $v(t) = A\ cos(\omega_0\ t) = 1.96 cos(1.58e10t)$ and $i_L(t) = 0.03 sin(1.58e10t)$. From Section 3 we have from Lemma 3.2, that for an LC oscillator with an odd-symmetric $i = f(v)$ function, $\omega_0 = \frac{1}{\sqrt{LC}} = 1.58e10$ rad/s which matches with the value from simulation. Also from the state equations we can obtain $i_L(t) = \frac{A}{\omega_0 L}\ sin(\omega_0 t) = 0.03 sin(1.58e10t)$ which again matches with the values from simulation. From Lemma 4.5 and with L=4 nH, C=1 pF and A= 1.96, we get

$$PPV(t) = \begin{pmatrix} -32.31 sin(1.58e10t) \\ 0.51 cos(1.58e10t) \end{pmatrix} \quad (18)$$

and from Lemma 5.2, we get $c = 521.84$ for N = 1. The results for the PPV of $v(t)$ and $i_L(t)$ obtained by our analytical formulae plotted against the results from numerical simulation for 61 Harmonic components case are shown in Fig.5 and for the 3 Harmonic case are shown in Fig.6. The phase noise constant c from numerical simulation for $N = 1$ was obtained as 515.83 for the case of 61 Harmonic components and as 521.86 for the case of 3 Harmonic components. We can observe that the results obtained from our formulae are in good agreement with the results from the numerical simulations for both the cases of 61 Harmonic components and 3 Harmonic components.

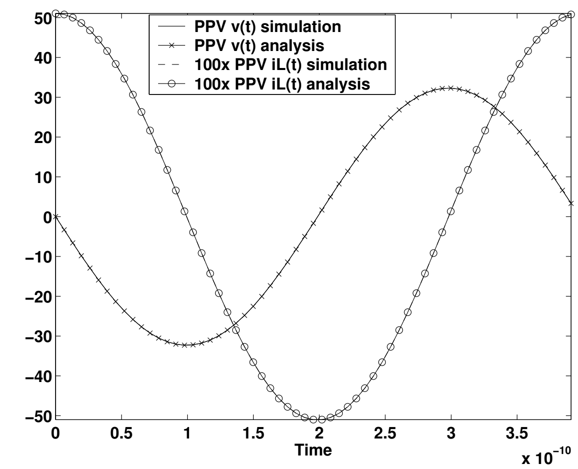

Figure 6: Simulation for PPV - 3 Components - CMOS oscillator

6.2 900 MHz, Q=4 tanh Oscillator

The oscillator used in this case is a LC oscillator circuit [3] whose $i = f(v)$ function is defined by a tanh curve. The i-v function for this circuit is given by $i = S tanh\left(\frac{G_n}{S} v\right)$ where $S = \frac{1}{R}$ and $-G_n > \frac{1}{R}$ ($= \frac{1.01}{R}$ for this oscillator) are the parameters that capture the negative-resistance mechanism that enables oscillations. The $i = f(v)$ function of this oscillator is odd-symmetric. The values of the circuit elements are L = 2 nH, C = 15 pF and the equivalent parallel resistance of the tank circuit R = 3 Ω.

The steady state solution of the circuit (Fig. 7) was obtained from a transient simulation as well as by Harmonic Balance as $v(t) = A cos(\omega_0 t) = 0.20 cos(5.77e9t)$ and $i_L(t) = 0.02 sin(5.77e9t)$. From Section 3 we have for an LC oscillator with a symmetric i-v nonlinearity, $\omega_0 = \frac{1}{\sqrt{LC}} = 5.77e9$ rad/s which matches with the value from simulation. Also from the state equations we can obtain $i_L(t) = \frac{A}{\omega_0 L}\ sin(\omega_0 t) = 0.02 sin(5.77e9t)$ which again matches with the values from simulation.

From Lemma 4.5 with L=2 nH, C=15 pF and A = 0.1985, we get

$$PPV(t) = \begin{pmatrix} -58.17 sin(5.77e9t) \\ 5.04 cos(5.77e9t) \end{pmatrix} \quad (19)$$

and from Lemma 5.2, we get $c = 1691.95$ for N = 1 The results for the PPV of $v(t)$ and $i_L(t)$ obtained by our analytical formulae plotted against the results from simulation for 61 Harmonic components case are shown in Fig. 8 and for the 3 Harmonic components case are shown in Fig 9. The phase noise constant c from simulation for $N = 1$ was obtained as 1695.35 for the case of 61 Harmonic components and as 1692.26 for the case of 3 Harmonic components. We can observe that the results obtained from our analytical formulae are in good agreement with the results obtained from the numerical simulations for both the 61 Harmonic and the 3 Harmonic cases of the simulation. *It is to be noted that our analytical formulae were developed assuming just the dc term and the first positive and negative harmonics for the state variables and the function $i(t)$. The excellent accuracy of our analytical results strongly validates the only assumption we made in Section 3.*

7 Conclusion

We identify three crucial parameters, derived from the incremental conductance of the oscillator's nonlinearity, that determine the

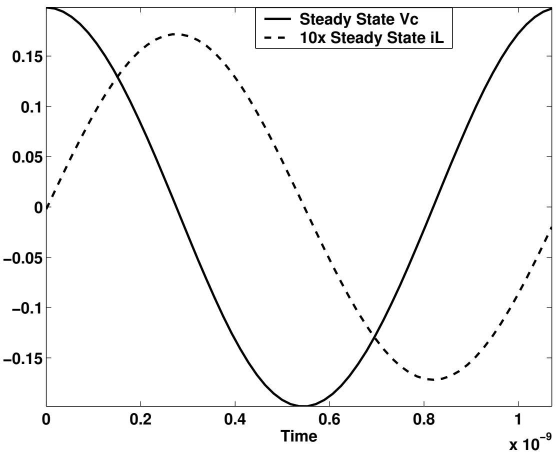

Figure 7: SS voltage $v(t)$, current $i_L(t)$ - tanh oscillator

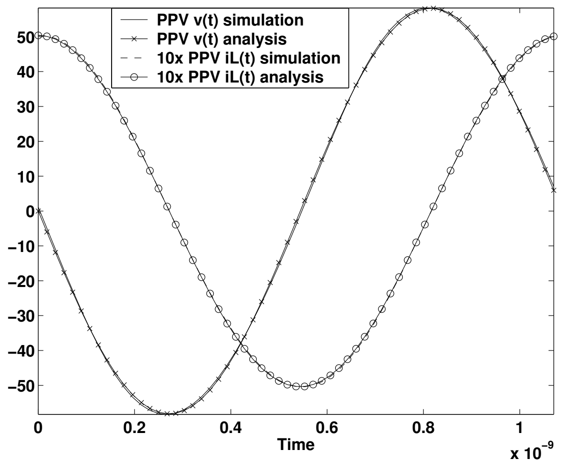

Figure 8: Simulation for PPV - 61 Components - tanh oscillator

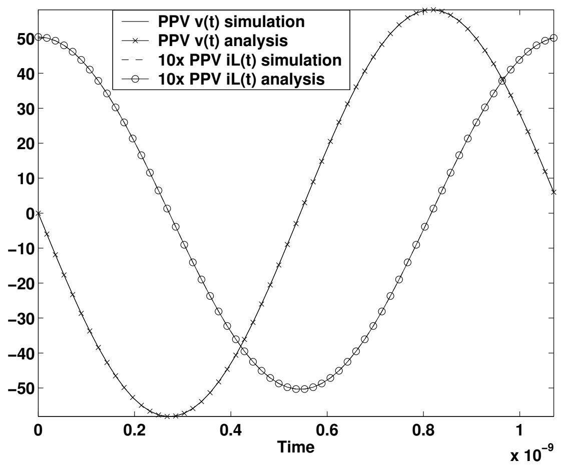

Figure 9: Simulation for PPV - 3 Components - tanh oscillator

phase noise and jitter characteristics of LC oscillators with arbitrary *i*-*v* nonlinearities. We derive accurate analytical expressions in terms of these parameters for phase noise eigenfunctions, phase noise power and timing jitter of generic LC oscillator structures. In doing so, we completely account for the nonlinear mechanism of the oscillator's dynamics. For the case of LC oscillators with symmetric *i*-*v* nonlinearities, we show that the phase noise power and jitter expressions take a simple form independent of the three parameters and show that our formulae match with the phase noise expressions from early analytical models. Comparison of our analytical formulae against numerical simulations demonstrates an excellent match.

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