Gen-Adler: The Generalized Adler's Equation for Injection Locking Analysis in Oscillators

Prateek Bhansali
ECE Department
University of Minnesota
Minneapolis, MN, 55455
e-mail: bhansali@umn.edu

Jaijeet Roychowdhury
ECE Department
University of Minnesota
Minneapolis, MN, 55455
e-mail: jr@umn.edu

Abstract— **Injection locking analysis based on classical Adler's equation is limited to LC oscillators as it is dependent on quality factor. In this paper, we present the Generalized Adler's equation applicable for injection locking analysis on oscillators independent of the circuit topology. The equation is obtained by averaging the PPV phase macromodel. The procedure is considerably simple and handy to determine the locking range for arbitrary shape small AC injection signal. Analytical equations for injection locking dynamics are formulated using the Generalized Adler's equation and validated with the PPV simulations.**

I. Introduction

Injection locking is a nonlinear phenomenon observed in oscillators. A small AC injection signal of frequency within the "locking range" entrains the oscillator at its frequency. The oscillator is thus said to be locked and the phenomenon is known as injection locking. Injection locking is exploited in the frequency synthesis and related applications. For example, it is used in the design of high performance quadrature oscillators [1], [2] and injection locked phase-locked loops (PLLs) [3]. Injection locking also has wide applications in optics [4], [5]. Thus, apart from its theoretical importance, study of injection locking is of great practical interest.

Injection locking has been widely studied for LC-tank based oscillators [6] using the classical Adler's equation [7]. Adler's equation provides quick insight into the locking dynamics. Furthermore, it provides rapid estimation of the locking range of oscillators for a given injection signal. The locking range is the frequency range around the oscillator's natural frequency for which injection locking occurs. But, Adler's equation is dependent on quality factor Q, which limits its use to LC oscillators. The Q factor by its definition is the ratio of energy preserved to energy dissipated per cycle in the oscillator. In LC oscillators, energy is stored in inductors and capacitors and a part of it is dissipated across the resistance. In a ring oscillator this definition does not hold good as there is no energy storage. There are no inductors in a ring oscillator and capacitances are charged and discharged within each oscillator cycle.

Recently, analytical techniques to predict injection locking range for ring oscillators have been presented in [8] and numerical techniques in [9] based on the Perturbation Projection Vector (PPV) [10]. These analytical and numerical techniques to obtain the locking range are accurate, being derived mathematically with no assumption of the circuit topology. But, they lack simplicity of Adler's equation which gives quick and handy graphical insight into the injection locking phenomenon. The alternate method is to study the injection locking using transient simulations. The transient analysis is time consuming as smaller time-steps are required for the accurate simulation of oscillators. The PPV based simulations are already established to be accurate and orders of magnitude faster over transient simulations in predicting injection locking [9]. Furthermore, it takes simulation for hundreds of cycles to conclusively identify the lock condition from the quasi-lock [6].

In this paper, we derive the Generalized Adler's (Gen-Adler's) equation from the PPV nonlinear phase macromodel. We apply an averaging technique on the PPV, to average out "fast" varying behavior and retain "slow" varying behavior, obtaining Gen-Adler's. Based on the analytical PPV of LC-tank oscillator [11], we illustrate the procedure to obtain the classical Adler's equation as a special case of Gen-Adler's equation. For an ideal ring oscillator, we formulate Gen-Adler's equation for sinusoidal, square and exponentially rising and falling AC injection signals based on its analytical PPV [12], giving useful insight into the injection locking phenomenon. The analytical formulation is validated against numerical computations and the locking dynamics are validated against already established PPV methods. It is worthwhile to emphasize that Gen-Adler's equation can be obtained for any oscillator with arbitrary AC injection signal of small amplitude numerically, but it is an approximation of the PPV.

This paper is organized as follows. In Section II, we briefly review Adler's equation and the PPV phase macromodel. Next, in Section III, we derive Gen-Adler's equation from the PPV by averaging it. In Section IV, we apply Gen-Adler's equation to analyze injection locking phase dynamics for both the LC tank oscillator and the ideal ring oscillator giving analytical expressions of the locking range. In Section V, we validate the formulation of Gen-Adler's and compare results with the PPV based simulations.

II. Background

In this section, we briefly review the classical Adler's equation [7] and the nonlinear PPV phase macromodel [10] for oscillators. Adler's equation is the analytical method proposed by Adler to predict injection locking. On the other hand, the PPV is a non-linear phase macromodel for oscillators, suitable for fast and accurate simulation of non-linear phenomenons in oscillators. We use the PPV to derive insightful Adler-like formulations for injection locking in oscillators.

A. Adler's Equation

In 1946, R. Adler obtained a differential equation known as Adler's equation for the oscillator's phase difference with the injection signal. This celebrated equation describes injection locking dynamics in LC oscillators. When a free running oscillator is perturbed by an AC injection signal, the lock dynamics are given by the following equation

$$\frac{d\Delta\phi(t)}{dt} = \Delta f_0 - \frac{V_i}{V}\frac{f_0}{2Q}\sin(\Delta\phi(t)) \tag{1}$$

where, $\Delta\phi(t)$ is the phase difference. The amplitude of the injected signal is V_i and the output amplitude of oscillator is V while the oscillator runs at a free running frequency of f_0. The frequency difference between the injected signal and the oscillator is Δf_0. Adler obtained injection locking behavior and the locking range for LC oscillators based on (1). When the oscillator is locked, the phase difference becomes constant and (1) gives

$$\frac{\Delta f_0}{f_0} = \frac{1}{2Q}\frac{V_i}{V}\sin(\Delta\phi(t)) \tag{2}$$

978-1-4244-2749-9/09/$25.00 ©2009 IEEE

and since $\sin(\cdot)$ lies between $+1$ and -1, we have

$$-\frac{1}{2Q}\frac{V_i}{V} \leq \frac{\Delta f_0}{f_0} \leq \frac{1}{2Q}\frac{V_i}{V} \tag{3}$$

This immediately gives the locking range f_L as

$$\begin{aligned} f_L &= 2|\Delta f_0|_{max} \\ f_L &= \frac{f_0}{Q}\frac{V_i}{V} \end{aligned} \tag{4}$$

Next, we review the PPV phase macromodel for oscillators which we will use to derive Adler-like equation. We start with the mathematical equation describing oscillator and list the phase deviation equation (7) under the effect of a perturbation signal [10].

B. The PPV phase macromodel

Let the oscillator has a state space $\vec{x}(t)$. The differential equation system describing it, under the effect of perturbation signal $\vec{b}(t)$, can be written as

$$\frac{d\vec{q}(\vec{x}(t))}{dt} + \vec{f}(\vec{x}) = \vec{b}(t) \tag{5}$$

where, $q(\cdot)$ and $f(\cdot)$ are nonlinear functions. If the perturbation signal is small then the amplitude variations can be neglected, and the solution $\vec{x}_p(t)$ of (5) will be

$$\vec{x}_p(t) = \vec{x}_s(t+\alpha(t)) \tag{6}$$

where, $\vec{x}_s(t)$ is the steady state T-periodic solution without any perturbation signal. The phase deviation or the time-shift in steady state solution , $\alpha(t)$, is given by a scalar equation [10]

$$\frac{d\alpha(t)}{dt} = \vec{v}_1^T(t+\alpha(t)) \cdot \vec{b}(t) \tag{7}$$

where, $\vec{v}_1^T$ is T-periodic and is called the PPV of the oscillator. The PPV has the same size as $\vec{x}(t)$. The elements of the PPV gives the phase sensitivity of the corresponding components of $\vec{x}(t)$ to an externally applied perturbation signal. The PPV can be calculated for oscillators numerically in the time and the frequency domain [13]. Interestingly, the analytical PPV for the LC-tank oscillator and the ideal ring oscillator exist in the literature [11], [12].

III. Generalized Adler's Equation

In this section, we use the PPV phase macromodel reviewed in the last section to obtain Generalized Adler's equation. We first change the variables in (7) from phase deviation $\alpha(t)$ to phase difference $\Delta\phi(t)$ to obtain a modified phase equation. We then perform averaging in the phase equation retaining the slow behavior to derive Gen-Adler's.

A. Modified Phase Equation

In this section, we derive the modified phase equation. Let $\vec{v}_1^T(t)$ be a PPV, periodic with frequency f_0. We can write

$$\vec{v}_1^T(t) = \vec{\chi}(f_0 t) \tag{8}$$

where, $\vec{\chi}(\cdot)$ is 1-periodic function *i.e.* $\chi(f_0 t) = \chi(f_0 t + 1)$.

Assuming the perturbation signal is periodic with frequency f_1, the PPV equation (7) is of the following form

$$\frac{d\alpha(t)}{dt} = \vec{\chi}(f_0(t+\alpha(t))) \cdot \vec{b}(f_1 t) \tag{9}$$

where, $b(\cdot)$ is 1-periodic function now. We can define phase difference between the perturbation signal and the oscillator as $\Delta\phi(t) = \phi(t) - \phi_1(t)$, for $\phi(t) = f_0(t+\alpha(t))$ and $\phi_1(t) = f_1 t$, to obtain

$$\begin{aligned} \Delta\phi(t) &= f_0(t+\alpha(t)) - f_1 t \\ \alpha(t) &= \frac{\Delta\phi(t)}{f_0} + \frac{f_1 - f_0}{f_0} t \end{aligned} \tag{10}$$

Next, differentiating (10) and substituting the value of $\dot{\alpha}(t)$ from (9), we get

$$\vec{\chi}(f_0(t+\alpha(t))) \cdot \vec{b}(f_1 t) = \frac{1}{f_0}\frac{d\Delta\phi(t)}{dt} + \frac{f_1 - f_0}{f_0} \tag{11}$$

Substituting the value of $\alpha(t)$ from (10) in (11), we obtain a modified phase equation suitable for further analysis

$$\frac{d\Delta\phi(t)}{dt} = -(f_1 - f_0) + f_0 \vec{\chi}((\Delta\phi(t) + \phi_1(t))) \cdot \vec{b}(\phi_1(t)) \tag{12}$$

In the modified phase equation (12), $\Delta\phi(t)$ is phase difference between the oscillator and the perturbation signal. For the injection locking analysis, perturbation signal is an AC injection signal with frequency close to the oscillator's natural frequency.

B. Gen-Adler's Equation

We now assume that in (12), $\phi_1(t)$ is "fast" varying and $\Delta\phi(t)$ "slowly" varying variable. This assumption is explained below in detail. Under this assumption, we can average out the fast $\phi_1(t)$ and retain the slow $\Delta\phi(t)$ variations in (12) to obtain $g(\Delta\phi(t))$ as

$$g(\Delta\phi(t)) = \frac{1}{T_1}\int_0^{T_1} \chi(\Delta\phi(t) + \phi_1(t)) \cdot b(\phi_1(t))\, d\phi_1(t) \tag{13}$$

where, $T_1 = \phi_1(\frac{1}{f_1})$. Hence,

$$g(\Delta\phi(t)) = \frac{1}{1}\int_0^{1} \chi(\Delta\phi(t) + \phi_1(t)) \cdot b(\phi_1(t))\, d\phi_1(t) \tag{14}$$

Therefore, after averaging out the fast variations, (12) can be written as

$$\boxed{\frac{d\Delta\phi(t)}{dt} = -(f_1 - f_0) + f_0 g(\Delta\phi(t))} \tag{15}$$

This is the Generalized Adler's equation valid for any oscillator as compared to the original Adler's equation (1), which is only applicable to LC-tank like oscillators. This is of same form as original Adler's equation for sinusoidal $g(\Delta\phi(t))$.

Under lock conditions, the phase difference between oscillator and injected signal becomes constant, that is $d\Delta\phi(t)/dt = 0$ or $\Delta\phi(t) = \Delta\phi_0$ and hence

$$\begin{aligned} f_1 - f_0 &= f_0 g(\Delta\phi_0) \\ \Delta f_0 &= f_0 g(\Delta\phi_0) \end{aligned} \tag{16}$$

Let f_L be the locking range of the oscillator. The maximum value of $g(\cdot)$ gives the locking range of the oscillator about f_0 as

$$\begin{aligned} |\Delta f_0|_{max} &= f_0\,[g(\Delta\phi(t))]_{max} \\ f_L &= 2|\Delta f_0|_{max} \end{aligned} \tag{17}$$

The locking range, f_L, is typically much smaller than f_1 or f_0. If the oscillator is not locked $\Delta\dot{\phi}(t) \neq 0$, and the maximum value of *RHS* in (15) is $-(f_1 - f_0) + f_L/2 \ll f_1$. Thus, we have

$$\begin{aligned} \left(\frac{d\Delta\phi(t)}{dt}\right)_{max} &= -(f_1 - f_0) + f_L/2 \\ \frac{d\phi_1(t)}{dt} &= f_1 \end{aligned} \tag{18}$$

As f_1 is close to free running frequency f_0 of the oscillator and f_L is small,

$$\left(\frac{d\Delta\phi(t)}{dt}\right)_{max} \ll \frac{d\phi_1(t)}{dt} \tag{19}$$

Hence, our assumption that $\phi_1(t)$ is fast varying and $\Delta\phi(t)$ is slowly varying is justified.

IV. Injection Locking Range using Gen-Adler's Equation

In this section, we apply the proposed equation in Section III on two oscillators, the LC oscillator and the ring oscillator. We obtain the analytical formulation of Gen-Adler's for both oscillators. To show the general applicability, we analyze the ring oscillator for various types of injection signals.

A. LC Oscillator

Consider a simple $-G_m$ LC tank oscillator as shown in Fig. 1. The state variables of this oscillator are voltage across the capacitor $v(t)$ and current through the inductor $i(t)$. The analytical PPV of the LC oscillator [11], when perturbed by a sinusoidal injection signal $b(t) = I_i \cos(2\pi f_1 t)$ as shown in Fig. 1, is given as

$$v_1^T(t) = -\sqrt{\frac{L}{C}}\frac{1}{A}\sin(2\pi f_0 t) \tag{20}$$

for a steady state solution of $v(t) = A\cos(2\pi f_0 t)$.

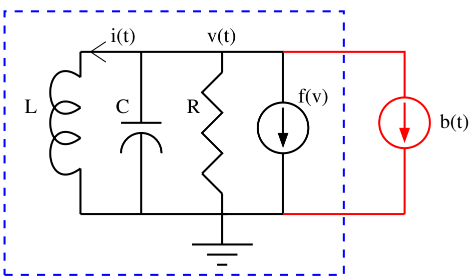


Fig. 1. LC tank oscillator circuit with injection signal $b(t)$.

The PPV can now be written in 1-periodic form using $\vec{v}_1^T(t) = \vec{\chi}(f_0 t)$ as

$$\begin{aligned} \chi(t) &= v_1^T(\frac{t}{f_0}) = -\sqrt{\frac{L}{C}}\frac{1}{A}\sin(2\pi t) \\ \chi(\Delta\phi(t) + \phi_1(t)) &= -\sqrt{\frac{L}{C}}\frac{1}{A}\sin(2\pi(\Delta\phi(t) + \phi_1(t))) \end{aligned} \tag{21}$$

To evaluate $g(\Delta\phi(t))$, we use $\chi(\cdot)$ from (21) in (14) and evaluate the integral

$$\begin{aligned} g(\Delta\phi(t)) &= \int_0^1 -\sqrt{\frac{L}{C}}\frac{1}{A}\sin(2\pi(\Delta\phi(t)+\phi_1(t)))\cdot\vec{b}(\phi_1(t))\,d\phi_1(t) \\ &= \int_0^1 -\sqrt{\frac{L}{C}}\frac{1}{A}\sin(2\pi(\Delta\phi+\phi_1))\cdot I_i\cos(\phi_1)\,d\phi_1 \\ &= -\frac{I_i}{2}\sqrt{\frac{L}{C}}\frac{1}{A}\sin(2\pi\Delta\phi) \end{aligned} \tag{22}$$

We note that the amplitude of the current through the inductor is given as $I_L = A\sqrt{\frac{C}{L}}$ and current through resistor as $I_R = QI_L$. Using this and plugging $g(\Delta\phi)$ in (15), we obtain the original Adler equation for the LC oscillator as

$$\frac{d\Delta\phi}{dt} = -(f_1 - f_0) - \frac{I_i}{I_R}\frac{f_0}{2Q}\sin(2\pi\Delta\phi) \tag{23}$$

This is a special case of Gen-Adler's equation, with a LC oscillator perturbed by a sinusoidal injection signal. As we show next, we can apply Gen-Adler's equation to study injection locking in ring oscillators directly unlike the original Adler's equation.

B. Ring Oscillator

In this subsection, we derive Gen-Adler's equation for the ring oscillator circuit shown in Fig. 2. All resistances and capacitors are assumed to be equal and inverters have ideal characteristics of the following form

$$f(V) = \begin{cases} A & \text{if } V < 0 \\ -A & \text{if } V \geq 0 \end{cases} \tag{24}$$

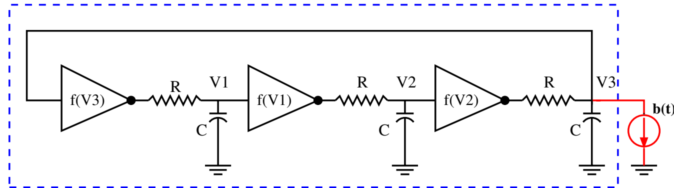


Fig. 2. Ring oscillator circuit with injection signal $b(t)$.

In [12], an analytical PPV for the node $V3$ of ideal ring oscillator was derived as

$$v_1^T(t) = \begin{cases} \frac{1}{\sqrt{5}}\frac{R}{A}e^{\frac{t}{\tau}} & \text{if } 0 \leq t < \frac{T}{2} \\ \frac{R}{A}\left(\frac{2}{\sqrt{5}} - 1\right)e^{\frac{t}{\tau}} & \text{if } \frac{T}{2} \leq t < T \end{cases} \tag{25}$$

where, τ is the RC time constant and $T = 2.88727\tau = 1/f_0$ is time period of the ring oscillator. To obtain the 1-periodic form of the PPV at node V3 we put $t = f_0 t$, which gives us

$$\begin{aligned} \chi(t) = v_1^T(t/f_0) &= \begin{cases} \frac{1}{\sqrt{5}}\frac{R}{A}e^{\frac{t}{f_0\tau}} & \text{if } 0 \leq t = \frac{t}{f_0} < \frac{T}{2} \\ \frac{R}{A}\left(\frac{2}{\sqrt{5}} - 1\right)e^{\frac{t}{f_0\tau}} & \text{if } \frac{T}{2} \leq t = \frac{t}{f_0} < T \end{cases} \\ &= \begin{cases} \frac{1}{\sqrt{5}}\frac{R}{A}e^{2.887t} & \text{if } 0 \leq t < \frac{1}{2} \\ \frac{R}{A}\left(\frac{2}{\sqrt{5}} - 1\right)e^{2.887t} & \text{if } \frac{1}{2} \leq t < 1 \end{cases} \end{aligned} \tag{26}$$

For notational simplicity,

$$\chi(t_s) = \begin{cases} K_1\frac{R}{A}e^{K_0 t_s} & \text{if } 0 \leq t_s < \frac{1}{2} \\ K_2\frac{R}{A}e^{K_0 t_s} & \text{if } \frac{1}{2} \leq t_s < 1 \end{cases} \tag{27}$$

where,

$$K_0 = 2.887, \quad K_1 = \frac{1}{\sqrt{5}}, \quad K_2 = \left(\frac{2}{\sqrt{5}} - 1\right) \tag{28}$$

Gen-Adler's equation for the ring oscillator is then obtained by evaluating $g(\Delta\phi(t)$ from (14) for each of the injection signals. After evaluating $g(\Delta\phi(t)$, we plot it for $R = 2k\Omega$, $C = 2nF$ and $A = 1V$ in each case. The lock range can be easily obtained from these graphs or by finding the maximum of $g(\Delta\phi(t)$. The injection current amplitude was taken to be $I_i = 0.1mA$ and the frequency, $f_1 = 1.02 f_0$, where f_0 is free running frequency of the ring oscillator.

1) Sinusoidal Injection Current: For a sinusoidal injection current signal $b(t) = I_i \sin(2\pi f_1 t)$ to the ring oscillator shown in Fig. 2, we have

$$b(\phi_1(t)) = I_i \sin(2\pi\phi_1(t)) \qquad \text{if } 0 \leq \phi_1 < 1 \tag{29}$$

where $\phi_1(t) = f_1 t$.

Therefore, from (14), (27), and (29)

$$g(\Delta\phi(t)) = \int_0^1 \chi(\Delta\phi(t) + \phi_1(t))\cdot b(\phi_1)\,d\phi_1 \tag{30}$$

So, $g(\Delta\phi)$ for the ring oscillator with sinusoidal injection signal is obtained as

$$\boxed{\begin{aligned} g(\Delta\phi(t)) = &\frac{1}{\sqrt{4\pi^2 + K_0^2}}\frac{RI_i}{A}\sin(2\pi\Delta\phi(t) + \zeta) \\ &\times\left[K_1(e^{K_0/2} + 1) - K_2(e^{K_0} + e^{K_0/2})\right] \end{aligned}} \tag{31}$$

where,

$$sin(\zeta) = \frac{2\pi}{\sqrt{4\pi^2 + K_0^2}}$$

and Gen-Adler's equation for the ring oscillator with sinusoidal injection is

$$\frac{d\Delta\phi}{dt} = -(f_1 - f_0) + f_0 g(\Delta\phi(t)) \tag{32}$$

The phase lock would occur, when $\frac{d\Delta\phi(t)}{dt}$ is 0 *i.e.*

$$\begin{aligned}\frac{(f_1 - f_0)}{f_0} &= \frac{1}{\sqrt{4\pi^2 + K_0^2}} \frac{RI_i}{A} \sin(2\pi\Delta\phi(t) + \zeta) \\ &\times \left[K_1(e^{K_0/2} + 1) - K_2(e^{K_0} + e^{K_0/2})\right]\end{aligned} \tag{33}$$

and the locking range is given at the maximum value of $g(\Delta\phi(t))$. In this case, the maximum value of $g(\Delta\phi(t))$ occurs when $\sin(2\pi\Delta\phi(t) + \zeta) = 1$, therefore

$$\begin{aligned}|\Delta f_0|_{max} &= \frac{f_0}{\sqrt{4\pi^2 + K_0^2}} \frac{RI_i}{A} \left[K_1(e^{K_0/2} + 1) - K_2(e^{K_0} + e^{K_0/2})\right] \\ &= 0.6773 f_0 \frac{RI_i}{A}\end{aligned} \tag{34}$$

and the locking range is $f_L = 2|\Delta f_0|_{max}$. This can be easily seen from the Fig. 3 graphically.

The steady state phase difference, $\Delta\phi_0$, between the oscillator and the injected signal is given by "stable" solution of (33). For $\Delta f_0/f_0 = 0.02$, Fig. 3 shows the plot of (33). It can be clearly seen from Fig. 3 that oscillator will lock to the injection signal only if $\Delta f_0/f_0$ line intersects $g(\Delta\phi(t))$, that is (33) has a valid solution. The line $\Delta f_0/f_0$ intersects $g(\Delta\phi(t))$ twice in a period. The two intersection points are marked as U (unstable) and S (stable). When $\Delta\phi(t)$ is perturbed slightly from the unstable point, it will move away from it towards a stable point, depending upon the sign of $d\Delta\phi(t)/dt$ as shown in Fig. 3. For given $g(\Delta\phi(t))$, the locking range is the maximum range of injection signal frequency, f_1, yielding a valid solution of (33).

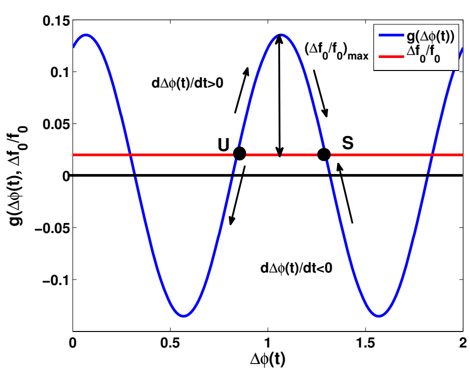


Fig. 3. Graphical solution of injection locking range in the ring oscillators with sinusoidal injection signal.

2) Square Wave Injection Signal: In Section IV-B.1, we derived Gen-Adler's equation for sinusoidal type of injection. In this section, we derive and analyze lock conditions for a square injection signal. For duty cycle η and a period of $T_1 = 1/f_1$, the characteristics of an ideal square wave are

$$b(t) = \begin{cases} I_i & \text{if } 0 \le t < \eta T_1 \\ 0 & \text{if } \eta T_1 \le t < T_1 \end{cases} \tag{35}$$

or in its 1-periodic form we have

$$b(\phi_1(t)) = \begin{cases} I_i & \text{if } 0 \le \phi_1(t) < \eta \\ 0 & \text{if } \eta \le \phi_1(t) < 1 \end{cases} \tag{36}$$

where, $\phi_1(t) = f_1 t$. Without loss of generality, we choose $0 < \eta \le 0.5$ and evaluate $g(\Delta\phi(t))$ as

$$g(\Delta\phi(t)) = \begin{cases} \frac{RI_i}{A}\frac{K_1}{K_0}\left[e^{K_0\Delta\phi(t)}(e^{\eta K_0} - 1)\right] \\ \quad \text{if } 0 \le \Delta\phi(t) < \frac{1}{2} - \eta \\ \frac{RI_i}{A}\frac{1}{K0}\left[(K_1 - K_2)e^{K_0/2} + e^{K_0\Delta\phi(t)}(K_2 e^{\eta K_0} - K_1)\right] \\ \quad \text{if } \frac{1}{2} - \eta \le \Delta\phi(t) < \frac{1}{2} \\ \frac{RI_i}{A}\frac{K2}{K0}\left[e^{K_0\Delta\phi(t)}(e^{\eta K_0} - 1)\right] \\ \quad \text{if } \frac{1}{2} \le \Delta\phi(t) < 1 - \eta \\ \frac{RI_i}{A}\frac{1}{K0}\left[(K_1 - K_2)e^{K_0/2} + e^{K_0\Delta\phi(t)}(K_2 - K_1 e^{(\eta-1)K_0})\right] \\ \quad \text{if } 1 - \eta \le \Delta\phi(t) < 1 \end{cases} \tag{37}$$

Similarly, $g(\Delta\phi(t))$ can be evaluated for $0.5 < \eta < 1$. To evaluate the locking range, we observe that $g(\Delta\phi(t))$ is composed of four piecewise exponential functions and is symmetric about zero. It is monotonically increasing for $0 \le \Delta\phi(t) < \frac{1}{2} - \eta$ and decreasing for $\frac{1}{2} - \eta \le \Delta\phi(t) < \frac{1}{2}$. Hence, the maximum value of $g(\Delta\phi(t))$ should occur at $\phi(t) = 1/2 - \eta$. Thus, the injection locking range for the ring oscillator with square wave injection signal is gives as

$$f_L = 2f_0 \frac{RI_i}{A}\frac{K_1}{K_0}\left[e^{K_0/2}(1 - e^{-\eta K_0})\right] \tag{38}$$

Fig. 4 shows the plot of (37) with the unstable and stable point for one period.

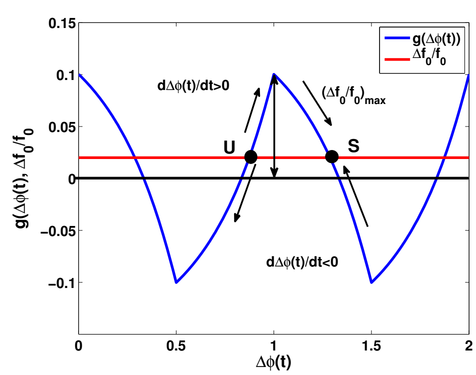


Fig. 4. Graphical solution of injection locking range in the ring oscillators with square injection signal, $\eta = 0.5$.

3) Exponentially rising-falling Wave Injection Signal: In this section, we apply an exponentially rising and falling injection signal and obtain Gen-Adler's equation for the ring oscillator. The exponentially rising falling wave can be represented as

$$b(t) = \begin{cases} I_i(1 - \varphi e^{K_0 t/T_1}) & \text{if } 0 \le t < T_1 \\ I_i(-1 + \varphi e^{-K_0(t - T_1/2)/T_1}) & \text{if } T_1 \le t < T_1 \end{cases} \tag{39}$$

or in its equivalent 1-periodic form as

$$b(\phi_1(t)) = \begin{cases} I_i(1 - \varphi e^{-K_0\phi_1(t)}) & \text{if } 0 \le 1 < \phi_1(t) \\ I_i(-1 + \varphi e^{-K_0(\phi_1(t) - 1/2)}) & \text{if } \phi_1(t) \le t < 1 \end{cases} \tag{40}$$

where, $\varphi = 1.6180339889$. Next, $g(\Delta\phi(t))$ is evaluated to obtain Gen-Adler's equation. For $0 \leq \Delta\phi(t) < 1/2$

$$\boxed{\begin{aligned} g(\Delta\phi(t)) = K_1 &\left[\frac{e^{K0/2} - 2e^{K_0\Delta\phi(t)} + 1}{K_0}\right] \\ - K_1 &\left[\varphi e^{K_0\Delta\phi(t)}\left(\frac{1}{2} - \Delta\phi(t) - \Delta\phi(t)e^{-k0/2}\right)\right] \end{aligned}} \tag{41}$$

and a simplified expression for $1/2 \leq \Delta\phi(t) < 1$ is as follows

$$\boxed{g(\Delta\phi(t)) = \frac{RI_i}{A}\left[(0.5283 - 0.4222\Delta\phi(t))e^{K_0\Delta\phi(t)} - 1.622\right]} \tag{42}$$

Gen-Adler's equation for an ideal ring oscillator with an exponential injection signal is given by (15), with $g(\Delta\phi(t))$ as obtained in (41) and (42).

To obtain the locking range of the ring oscillator in this case, we calculate the maximum value of $g(\Delta\phi(t))$ by differentiating it. The maximum value occurs at $\Delta\phi(t) = 0.9050$ and the locking range is given as

$$f_L = 0.744 f_0 \frac{RI_i}{A} \tag{43}$$

The plot of $g(\Delta\phi(t))$ and $\Delta f_0/f_0$ is shown in Fig. 5, which also gives insight into the locking.

Thus, we have derived Gen-Adler's equation for the ring oscillator with different injection signals. The locking range was calculated by finding the maximum of $g(\Delta\phi(t))$ and also graphically by plotting the Gen-Adler's equation. The locking range is directly proportional to the resistance R, the injection current amplitude I_i and inversely proportional to the amplitude of ring oscillator's output voltage A.

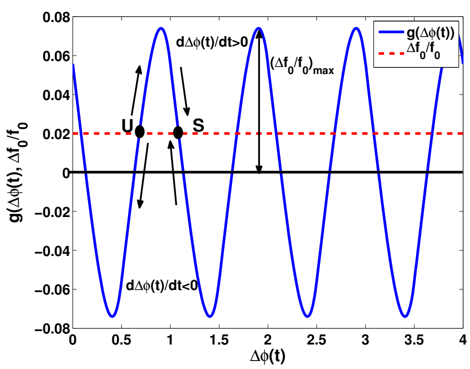


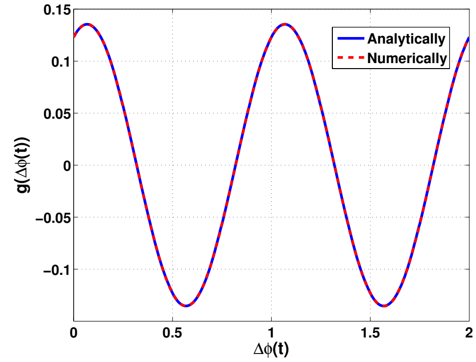
Fig. 5. Graphical solution of injection locking range in the ring oscillator with exponential injection signal.

V. Validation

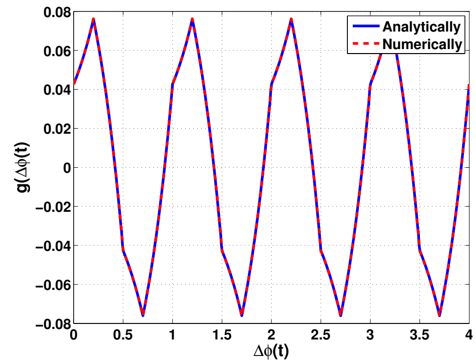
In this section, we first validate the formulation of Gen-Adler's equation by numerically computing $g(\Delta\phi(t))$ and comparing with analytical equations obtained in Section IV-B. Next, we plot the injection locking dynamics obtained using Gen-Adler's together with the PPV based simulations.

A. Numerical computation of $g(\Delta\phi)$

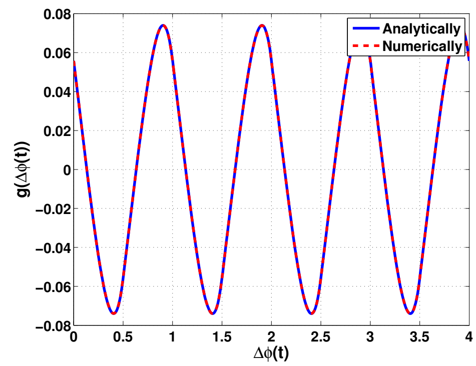
The function $g(\Delta\phi)$ was computed numerically in Matlab using the trapezoidal quadrature rule for function definitions of the PPV of the ring oscillator and different injection signals. The plots comparing analytical results with numerical computations are shown in Fig. 6. The excellent match validates the analytical formulation of Gen-Adler's.



(a) Sinusoidal injection signal



(b) Square wave injection signal, $\eta = 0.3$



(c) Exponential injection signal

Fig. 6. Numerical computation of $g(\Delta\phi)$ for the ring oscillator

B. Comparison with PPV based simulation

In this section, we plot injection locking dynamics of the ring oscillator using Gen-Adler's equation. The results obtained are compared with the PPV simulations for same circuit parameters. The phase difference obtained using Gen-Adler's is termed as $\Delta\phi_{Gen-Adler}$ in the locking dynamics plots Fig. 7, Fig. 8 and Fig. 9. The phase deviation, $\alpha(t)$, is obtained using (7). The phase difference $\Delta\phi_{PPV}(t)$ is then calculated using (10) as

$$\Delta\phi_{PPV}(t) = f_0 t - f_1 t + f_0\alpha(t) \tag{44}$$

For a square wave injection signal with duty cycle $\eta = 0.3$ and frequency $f_1 = 1.02 f_0$, the locking dynamics are shown in Fig. 8 for $t = 80/f_0$. We start with two different values of the initial phase difference, $\Delta\phi(0) = 1$ and $\Delta\phi(0) = 1.8$. As it can be clearly seen from Fig. 8 that the simulations based on Gen-Adler's matches with the PPV simulations for same initial conditions. For example, starting with $\Delta\phi(0) = 1$, the phase difference converges to $\Delta\phi(t) = 1.37$ in similar fashion for both, the PPV and the Gen-Adler's. The phase difference $\Delta\phi_0$ when oscillator is locked to the injection signal can also be seen from Fig. 4. For example, starting at the initial phase difference $\Delta\phi(0) = 1$, Fig. 4 predicts the phase difference between the ring oscillator and the injection signal to be $\Delta\phi = 1.3735$, when in lock.

Similarly, we get close match between the transient behavior obtained using Gen-Adler's equation and the PPV based simulation for sinusoidal and exponential injection signals, as shown in Fig. 7 and Fig. 9. However, it must be noted that PPV is able to capture the strong nonlinearities in the locking dynamics waveform and is more accurate than Gen-Adler's equation. On the other hand, Gen-Adler's is useful to evaluate the locking range of oscillators and get quick insight into it.

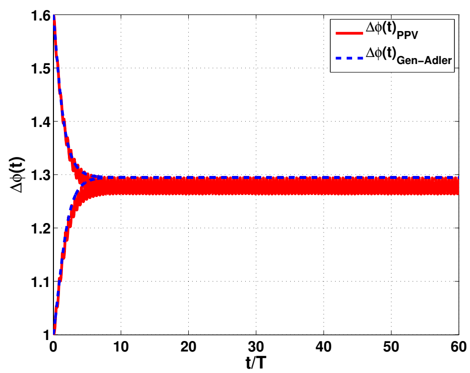


Fig. 7. Transient behavior in the ring oscillator injection locking with the sinusoidal injection signal.

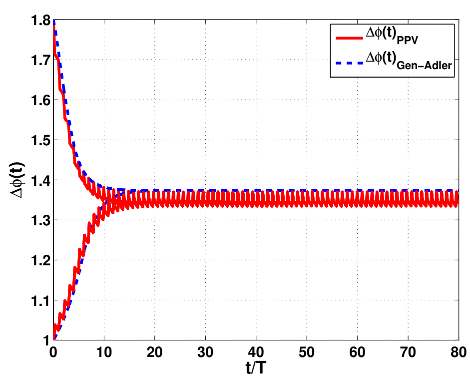


Fig. 8. Transient behavior in the ring oscillator injection locking with the square wave injection signal, $\eta = 0.3$.

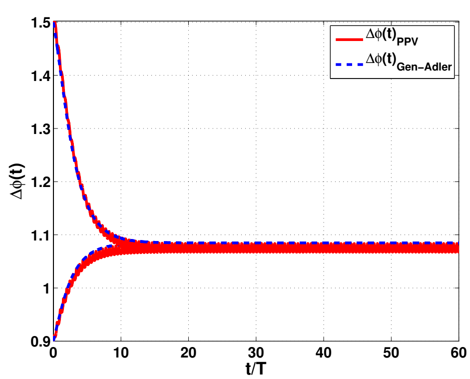


Fig. 9. Transient behavior in the ring oscillator injection locking with the exponential injection signal.

VI. Conclusions

In this paper, we have presented a generalized equation for injection locking analysis, Gen-Adler's. The equation provides insight into injection locking range and analytical formulations helps in quick injection locking analysis. The equation is derived as an approximation of the accurate PPV phase macromodel. We have formulated analytical lock dynamics equations and locking range formulae for the ring oscillator with sinusoidal, square and exponential injection signals.

Acknowledgments

We thank the National Science Foundation, the Semiconductor Research Corporation and the Gigascale Systems Research Center which supported this work.

References

[1] P. Kinget, R. Melville, D. Long, and V. Gopinathan. An injection-locking scheme for precision quadrature generation. *IEEE Journal of Solid-State Circuits*, 37(7):845–851, July 2002.

[2] C.J.M. Verhoeven. A high-frequency electronically tunable quadrature oscillator. *IEEE Journal of Solid-State Circuits*, 27(7):1097–1100, July 1992.

[3] S. Kudszus, M. Neumann, T. Berceli, and W.H. Haydl. Fully integrated 94-ghz subharmonic injection-locked pll circuit. *IEEE Transactions on Microwave and Guided Wave Letters*, 10(2):70–72, February 2000.

[4] A. E. Seigman. *Lasers*. University Science Books, 1986.

[5] J. Dorring, Li Yan, D.A. Satorius, and Y.J. Chen. Injection locking of an additive-pulse mode-locked fiber laser. *IEEE Photonics Technology Letters*, 14(11):1497–1499, November 2002.

[6] B. Razavi. A study of injection pulling and locking in oscillators. *IEEE Journal of Solid-State Circuits*, 39(9), September 2004.

[7] R. Adler. A study of locking phenomena in oscillators. *Proceedings of the I.R.E. and Waves and Electrons*, 34:351–357, June 1946.

[8] X. Lai and J. Roychowdhury. Analytical equation for predicting injection locking in lc and ring oscillators. *Proc. IEEE CICC*, September 2005.

[9] X. Lai and J. Roychowdhury. Capturing injection locking via nonlinear phase domain macromodels. *IEEE Transactions on Microwave Theory and Techniques*, 52(9):2251–2261, September 2004.

[10] A. Demir, A. Mehrotra, and J. Roychowdhury. Phase noise in oscillators: a unifying theory and numerical methods for characterization. *IEEE Trans. on Circuits and Systems-I:Fundamental Theory and Applications*, 47(5):655–674, May 2000.

[11] P. Ghanta, Z. Li, and J. Roychowdhury. Analytical Expressions for Phase Noise Eigenfunctions of LC Oscillators. In *Proc. IEEE ASP-DAC*, pages 175–180, January 2004.

[12] S. Srivastava and J. Roychowdhury. Analytical equations for nonlinear phase errors and jitter in ring oscillators. *IEEE Trans. on Circuits and Systems-I:Fundamental Theory and Applications*, 54(10):2321–2329, October 2007.

[13] A. Demir and J. Roychowdhury. A Reliable and Efficient Procedure for Oscillator PPV Computation, with Phase Noise Macromodelling Applications. *IEEE Trans. on Circuits and Systems-I:Fundamental Theory and Applications*, pages 188–197, February 2003.