Manifold Construction and Parameterization for Nonlinear Manifold-Based Model Reduction

Chenjie Gu and Jaijeet Roychowdhury
{gcj,jr}@eecs.berkeley.edu

University of California, Berkeley

Outline

- **Background**
  - Introduction to MOR and maniMOR
  - Manifold construction and parameterization
- **Manifold construction using integral curves**
  - DC manifold and the normalized integral curve equation
  - Ideal and almost-ideal manifold
  - Algorithm
- **Experimental results**
- **Conclusion**

Background

Model Order Reduction

Original system (size n)

$\vec{u}(t)$ → $$\frac{d\vec{x}}{dt} = f(\vec{x}) + B\vec{u}(t)$$ → $\vec{y} = C\vec{x}$

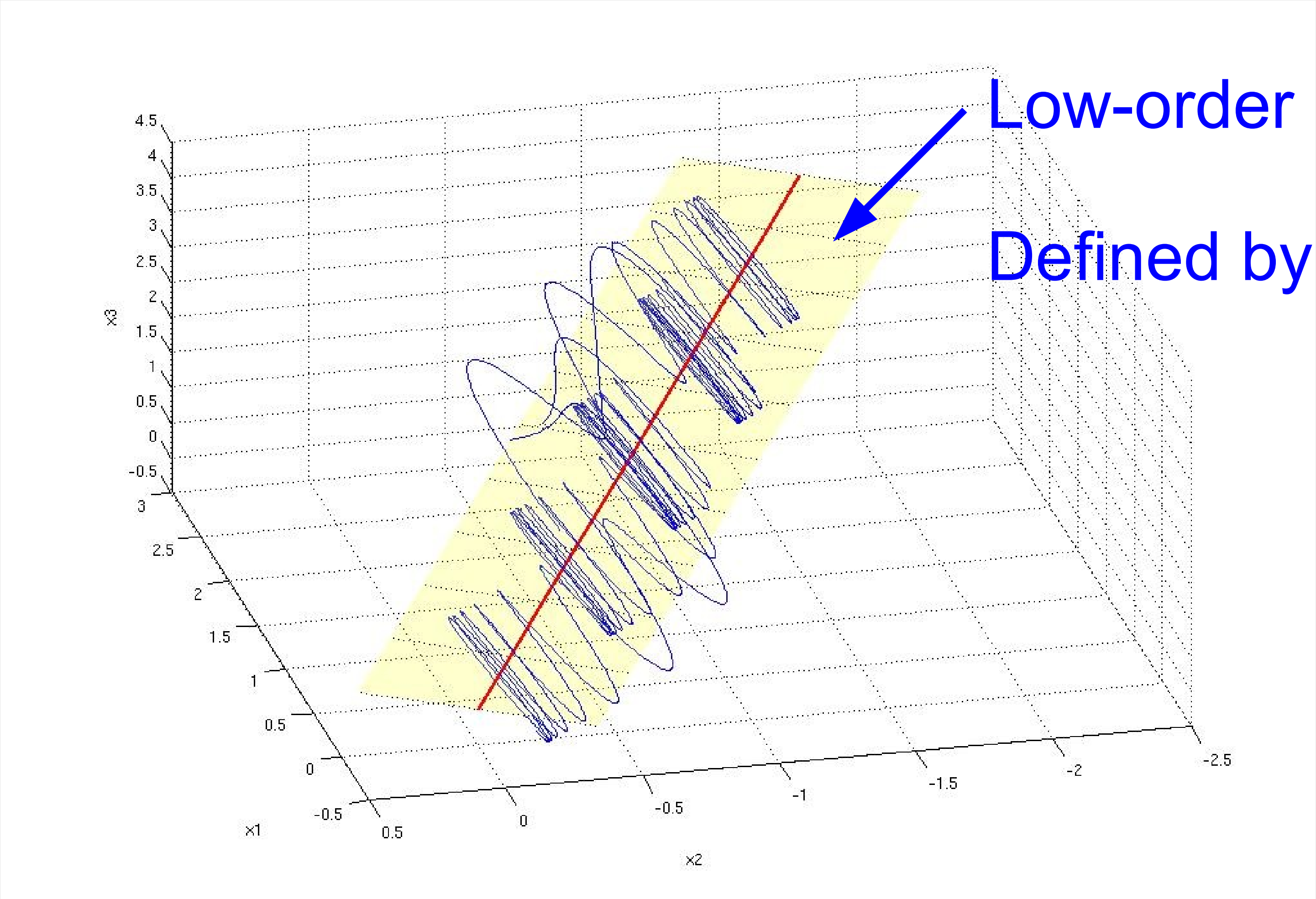
$$v : \vec{x} \mapsto \vec{z}, \qquad \vec{x} \in \mathbb{R}^n, \vec{z} \in \mathbb{R}^q, \qquad q \ll n$$

Reduced system (size q)

$\vec{u}(t)$ → $$\frac{d\vec{z}}{dt} = f_r(\vec{z}) + B_r\vec{u}(t)$$ → $\vec{y} = C_r\vec{z}$

Low-order Linear Subspace

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

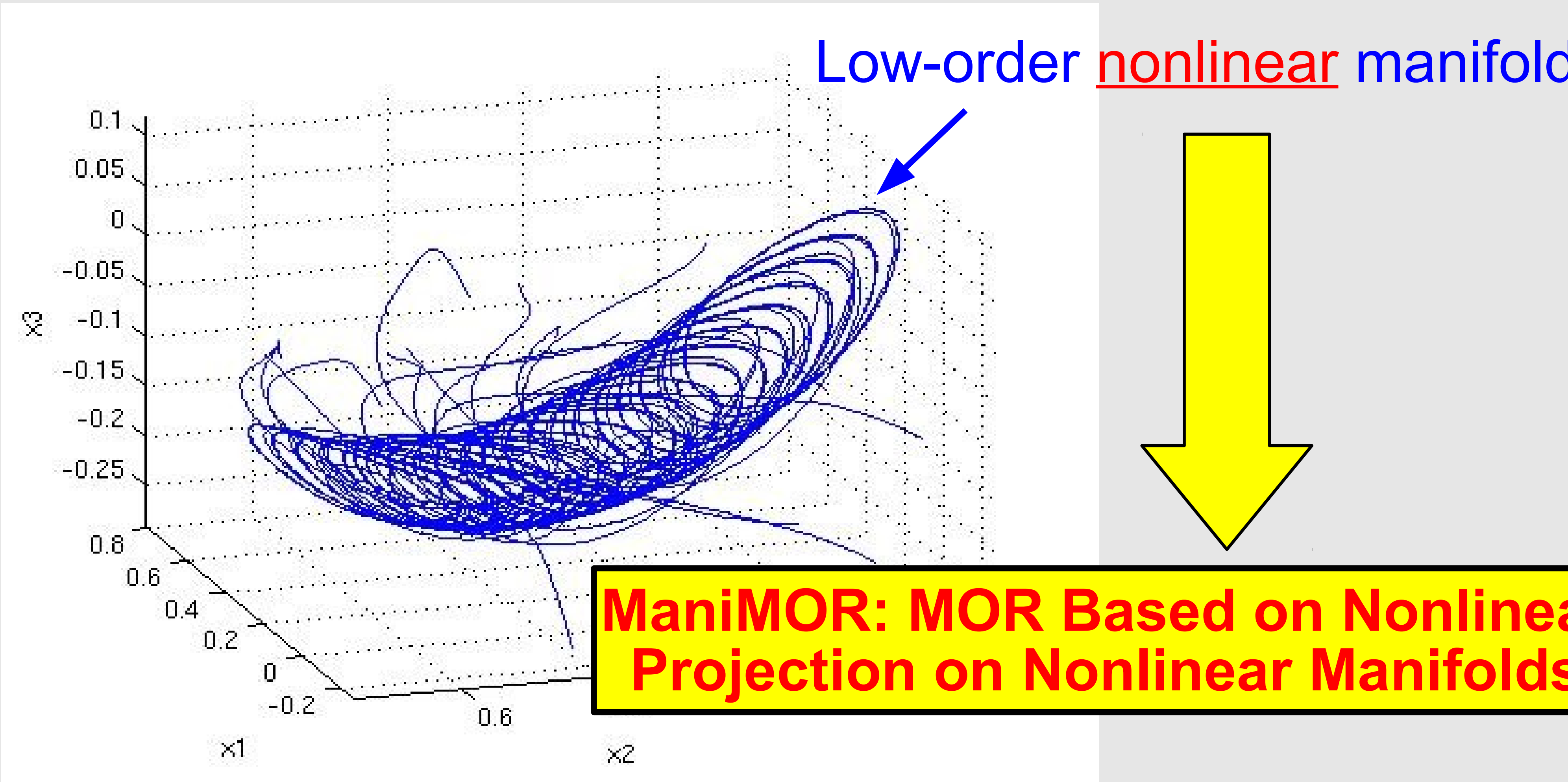


Low-order linear subspace

Defined by $x = Vz$

Low-order Nonlinear Manifold

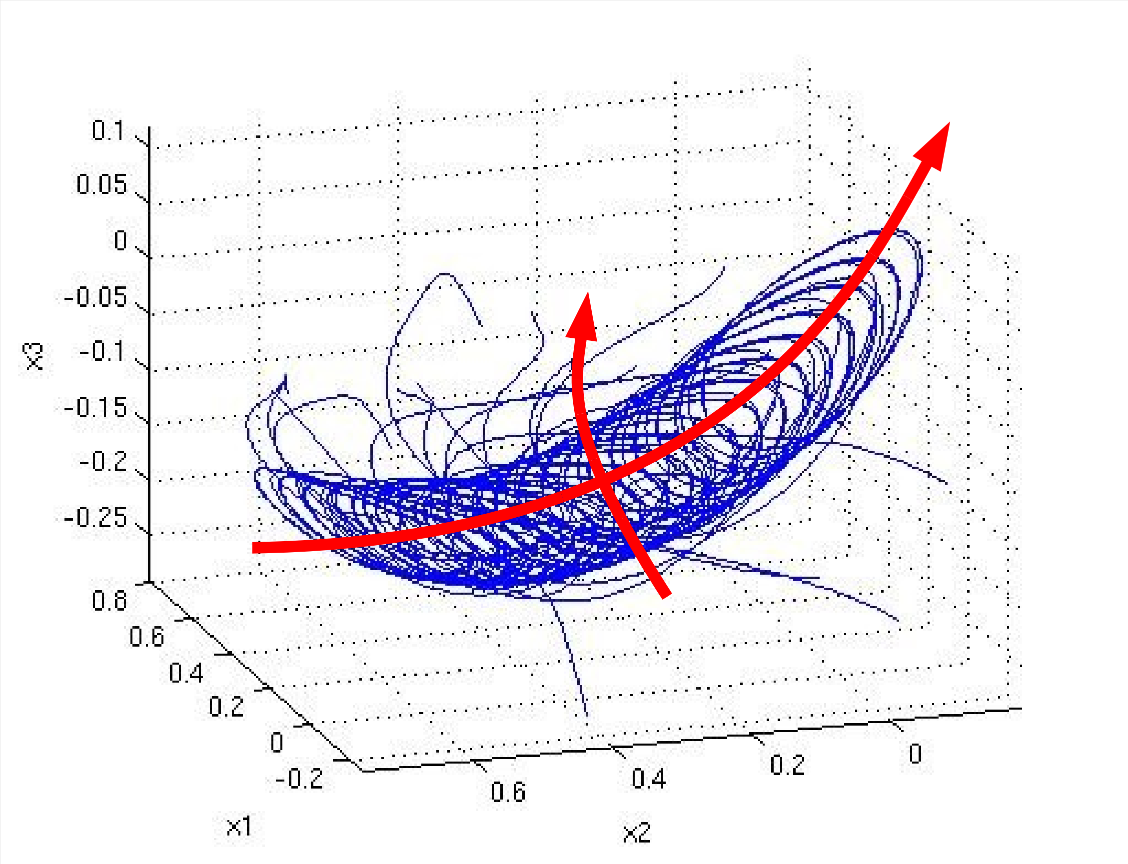
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ x_1^2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$



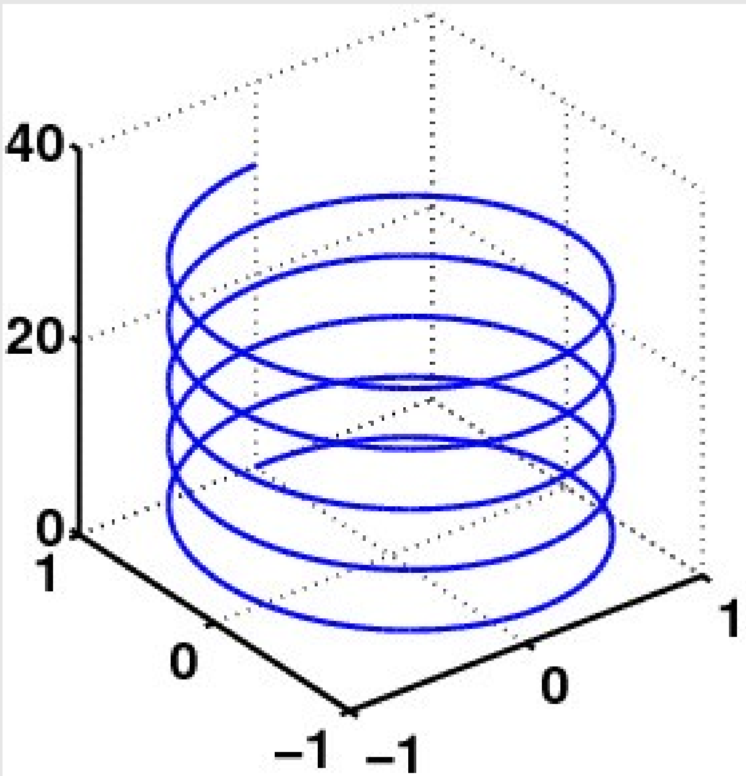
ManiMOR: MOR Based on Nonlinear Projection on Nonlinear Manifolds

Key Steps in ManiMOR

- **"Find" the nonlinear manifold**
  - Capture important dynamics
- **"Parameterize" the manifold**
  - Build up the coordinate system

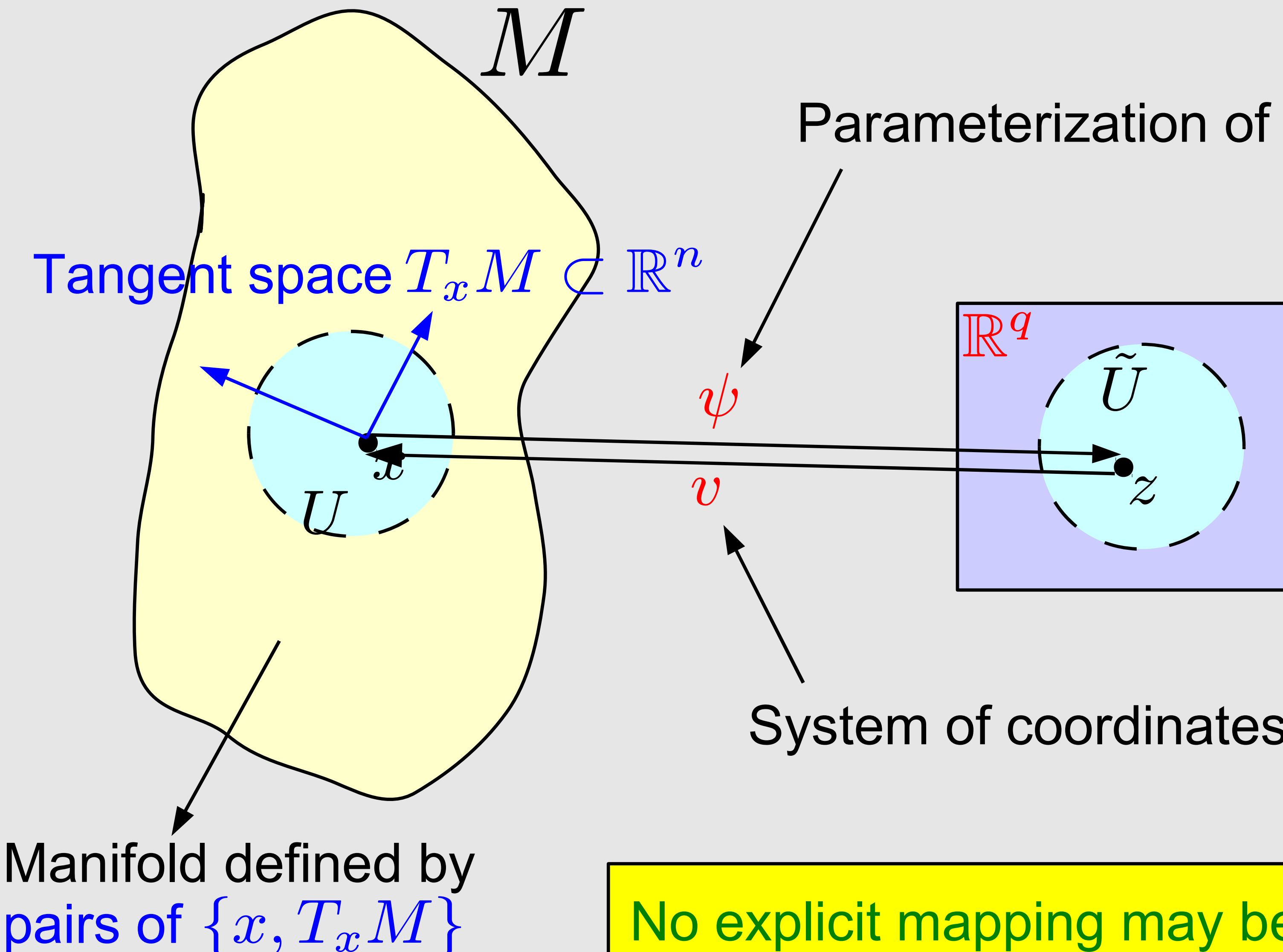


Manifold and Its Parameterization



$$\begin{cases} x = \cos(t) \\ y = \sin(t) \\ z = t \end{cases}$$

Manifold and Its Parameterization

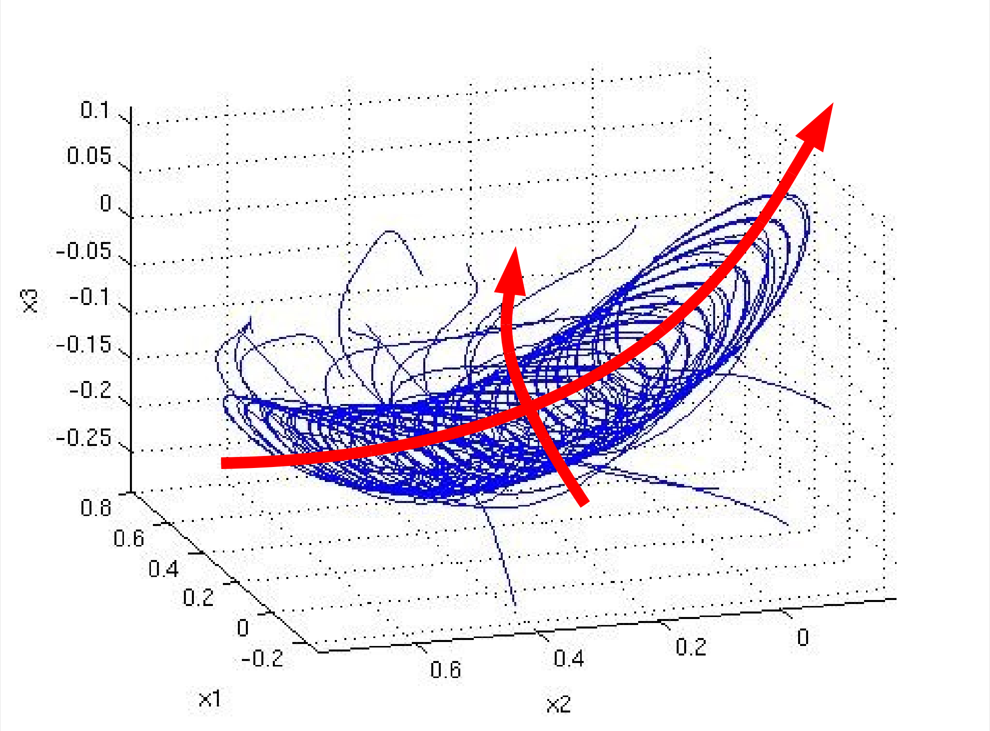


No explicit mapping may be derived.
Instead, use piecewise linear approximation.

Manifold and Its Parameterization

1. Identify the manifold
   that capture important dynamics

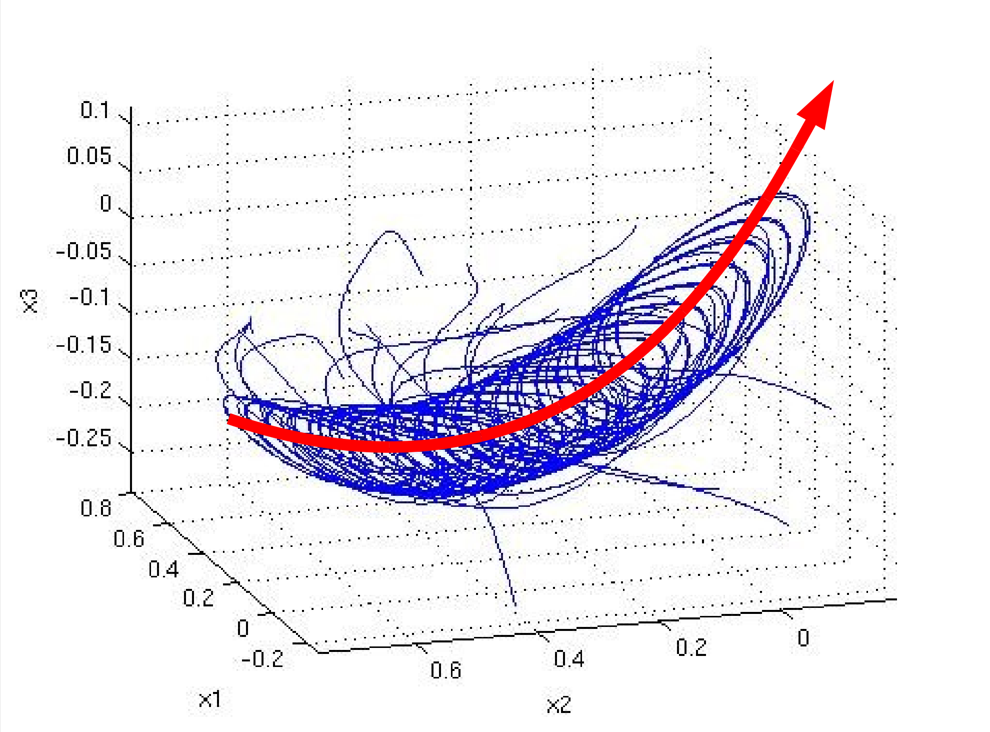
2. Compute and store pairs of $\{x, T_x M\} / \{z, T_z M\}$



DC Manifold

DC operating points constitute a DC manifold.

$$\frac{d\vec{x}}{dt} = f(\vec{x}) + B\vec{u}(t) = 0$$



How to **compute** and **parameterize** the DC manifold?

DC Manifold

$$f(\vec{x}) + B\vec{u}(t) = 0$$

A straight-forward solution:

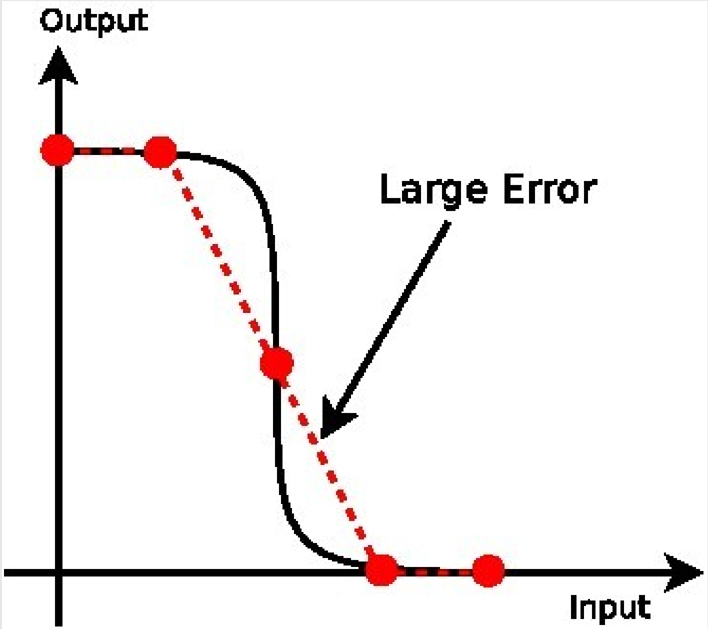
Computation: Perform DC sweep analysis

Parameterization: Define z coordinates using values of u

Problems:

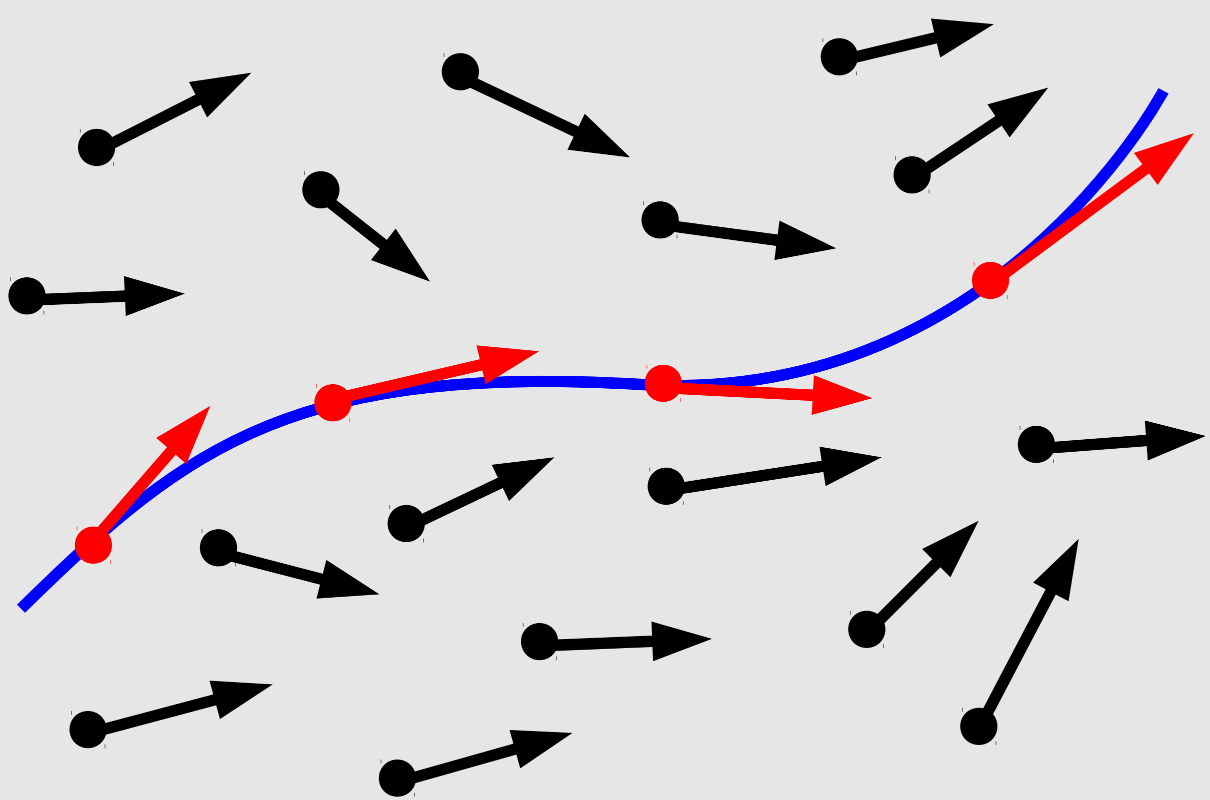
Hard to choose step size in DC sweep analysis

Not generalizable to higher dimensions



Introduction to Integral Curve

Given a vector field $v(x)$,
its integral curve is the curve $\gamma \equiv x(t)$ such that $\frac{dx}{dt} = v(x)$



DC Manifold as an Integral Curve

Need to derive the relationship between dx and du

$$f(\vec{x}) + B\vec{u}(t) = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial x}\frac{dx}{du} + B = 0$$

$$\frac{dx}{du} = -[G(x)]^{-1}B$$

The first Krylov basis.

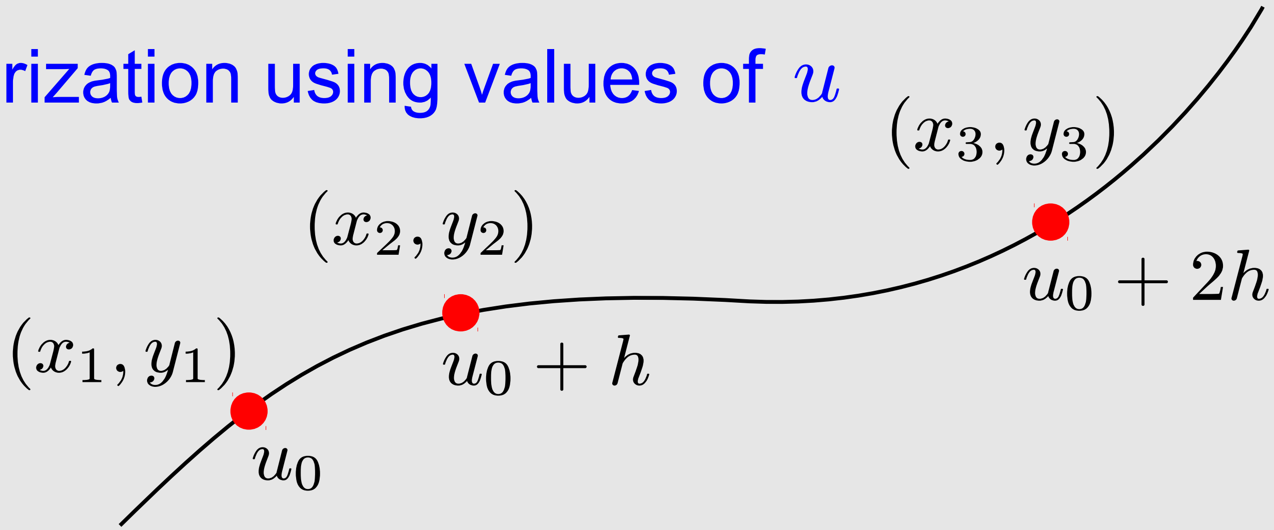
Initial condition: $x(u=0) = x_{DC}|_{u=0}$

Solutions are DC operating points.

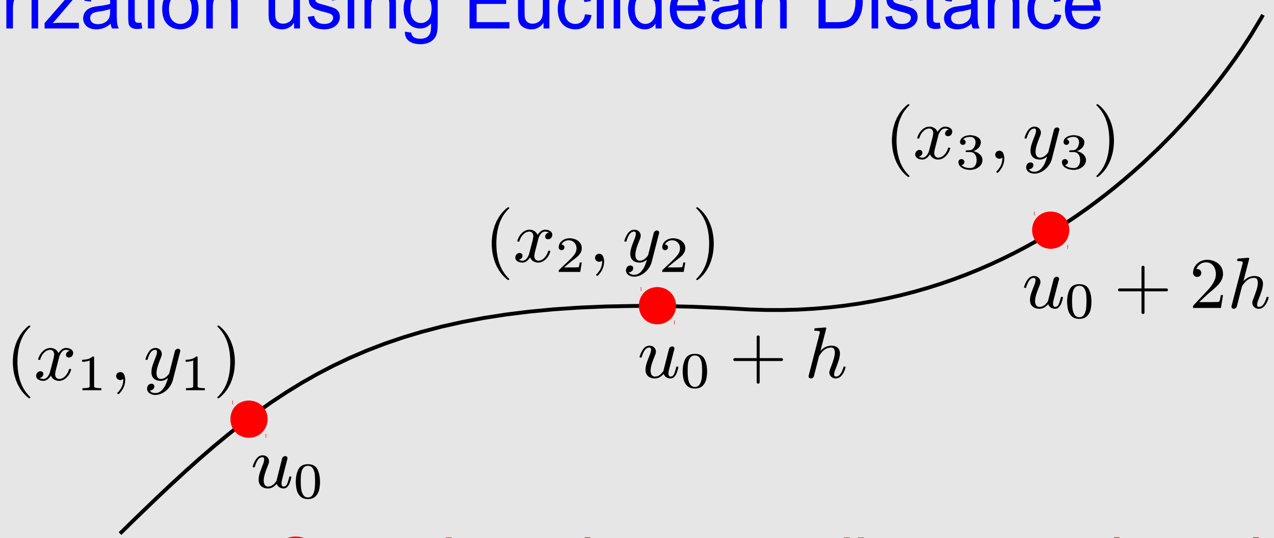
Any numerical integration / transient analysis code can be applied.

Parameterization using Euclidean Distance

Parameterization using values of u

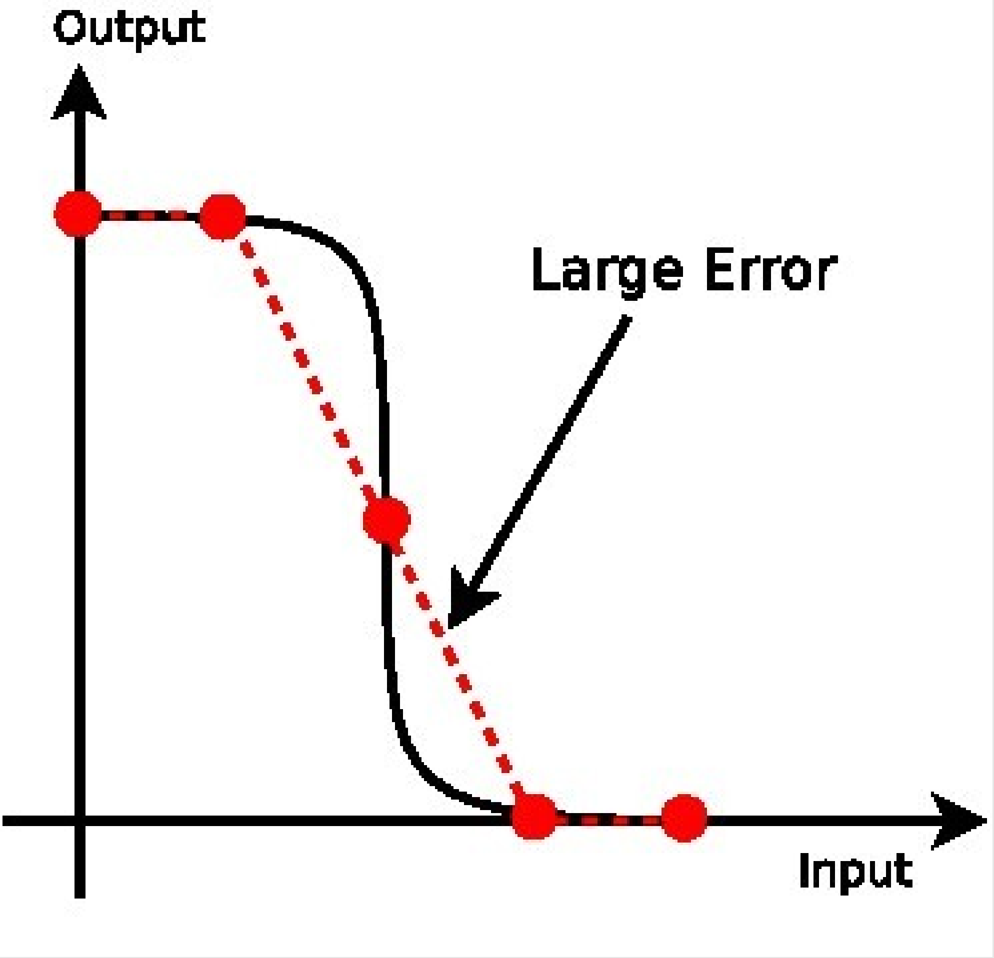


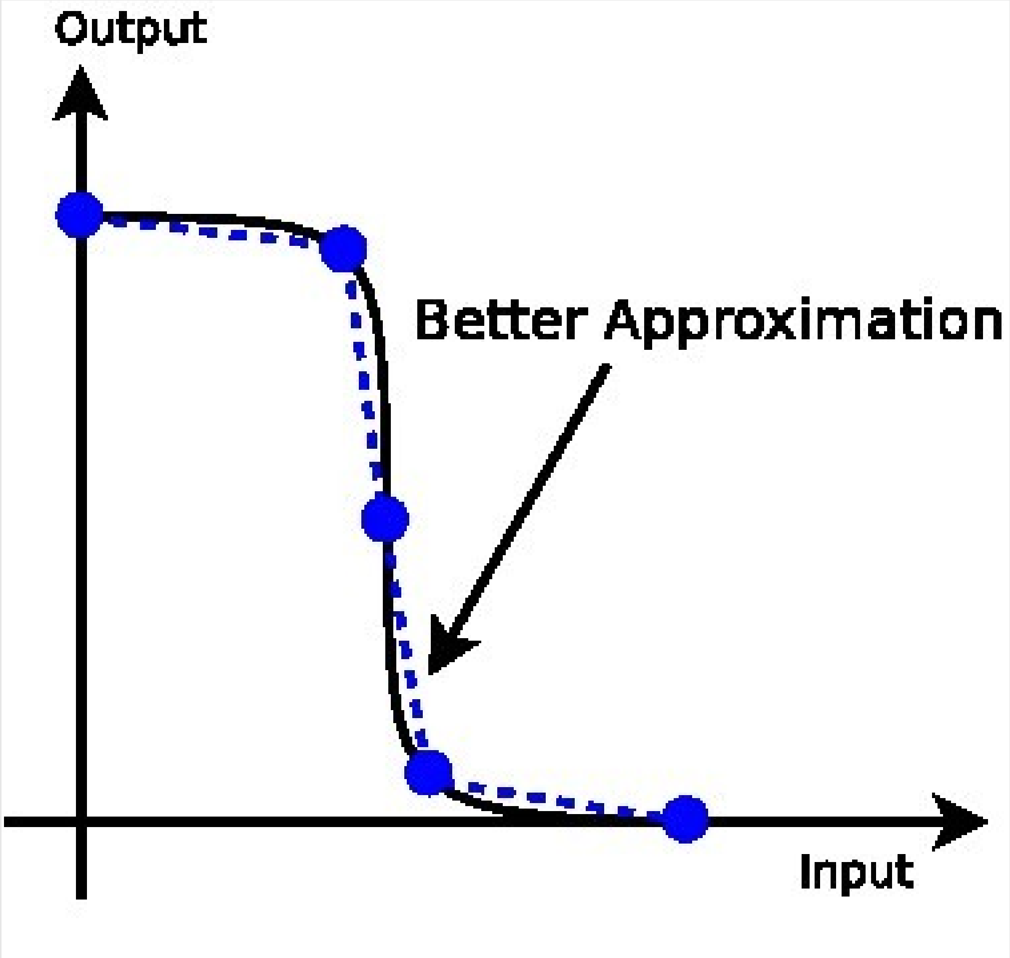
Parameterization using Euclidean Distance



Sample points equally spaced on the DC manifold

Parameterization using Euclidean Distance





Normalized Integral Curve Equation

Local Euclidean distance is

$$||dx||_2 = |du|$$

$$\frac{dx}{du} = -[G(x)]^{-1}B$$

$$\left|\left|\frac{dx}{du}\right|\right|_2 = ||[G(x)]^{-1}B||_2 = 1$$

Generally not satisfied

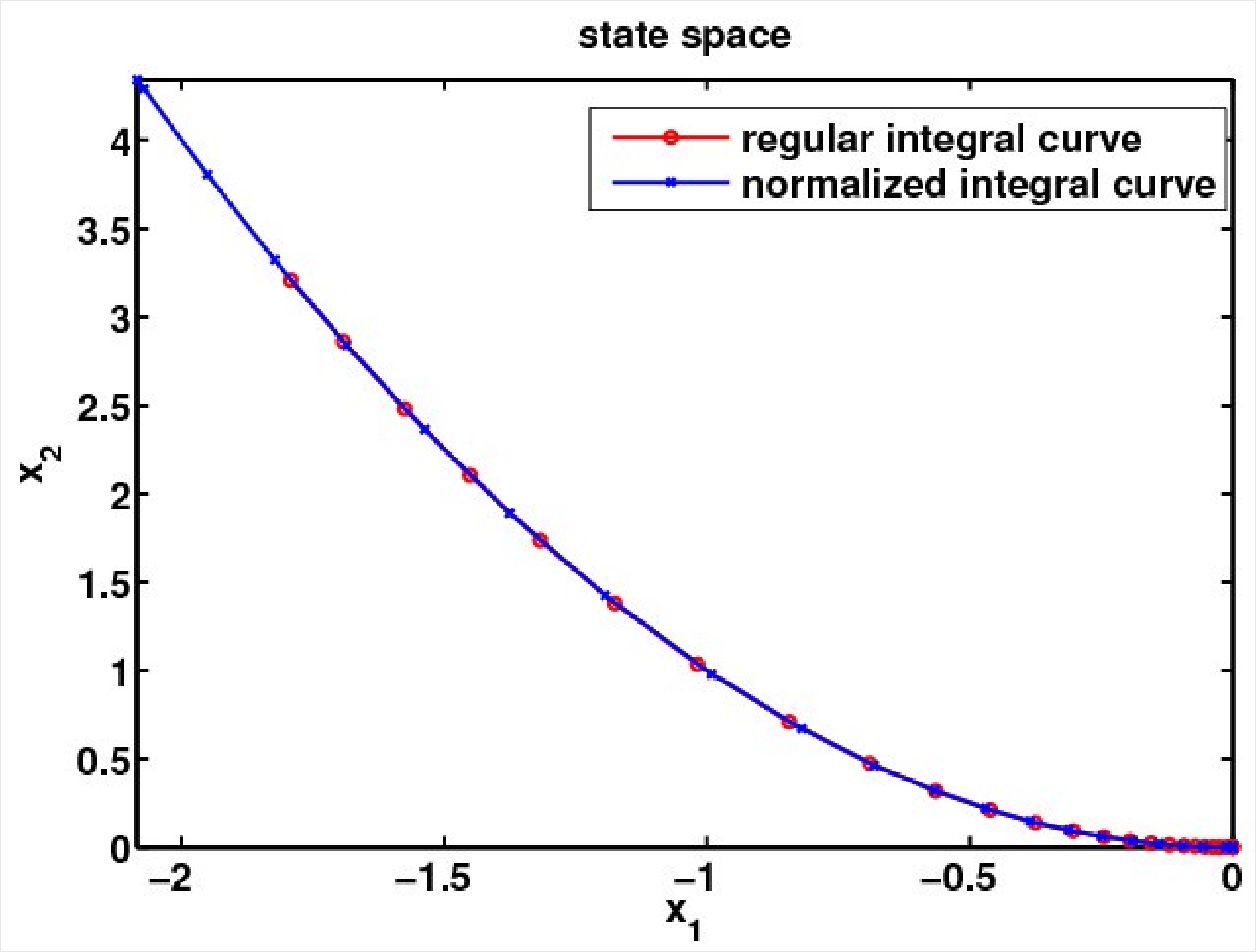
Normalize RHS

$$\frac{dx}{du} = \frac{[G(x)]^{-1}B}{||[G(x)]^{-1}B||_2}$$

Normalized Integral Curve Equation

Does it define the same integral curve?

Validation


state space
regular integral curve
normalized integral curve
4
3.5
3
2.5
2
1.5
1
0.5
0
−2
−1.5
−1
−0.5
0
x_1
x_2

Normalized Integral Curve Equation

Theorem:

Suppose $t = \sigma(\tau)$; $x(t)$ and $\hat{x}(\tau)$ satisfy

$$\frac{d}{dt}x(t) = g(x(t)) \text{ and } \frac{d}{d\tau}\hat{x}(\tau) = \sigma'(\tau)g(\hat{x}(\tau))$$, respectively.

Then $x(t)$ and $\hat{x}(\tau)$ span the same state space.

Sketch of proof:

Since $t = \sigma(\tau)$, we have $dt = \sigma'(\tau)d\tau$.

Define $\hat{x}(\tau) \equiv x(t) = \hat{x}(\sigma(t))$, then

$$\frac{d}{d\tau}\hat{x}(\tau) = \frac{d\hat{x}(\tau)}{dt}\frac{dt}{d\tau} = \sigma'(\tau)g(x(t)) = \sigma'(\tau)g(\hat{x}(\tau))$$

Normalized Integral Curve Equation

$$\frac{dx}{du} = -[G(x)]^{-1}B$$

$$\frac{dx}{du} = \frac{[G(x)]^{-1}B}{||[G(x)]^{-1}B||_2}$$

Solution: $x(u)$

Solution: $\hat{x}(\hat{u})$

Define $u = \sigma(\hat{u}) = \int_0^{\hat{u}} \frac{1}{||[G(\hat{x}(\mu))]^{-1}B||_2} d\mu$

From the theorem,
$x(u)$ and $\hat{x}(\hat{u})$ define the same integral curve.

Normalized Integral Curve Equation

$$\frac{dx}{du} = \frac{[G(x)]^{-1}B}{||[G(x)]^{-1}B||_2}$$

The first <u>normalized</u> Krylov basis.

Directly available from Krylov subspace methods.

Generalizable to higher dimensions.

Ideal Nonlinear Manifold

$$\frac{\partial x}{\partial z_1} = v_1(x), \qquad \frac{\partial x}{\partial z_2} = v_2(x), \qquad \cdots, \qquad \frac{\partial x}{\partial z_q} = v_q(x).$$

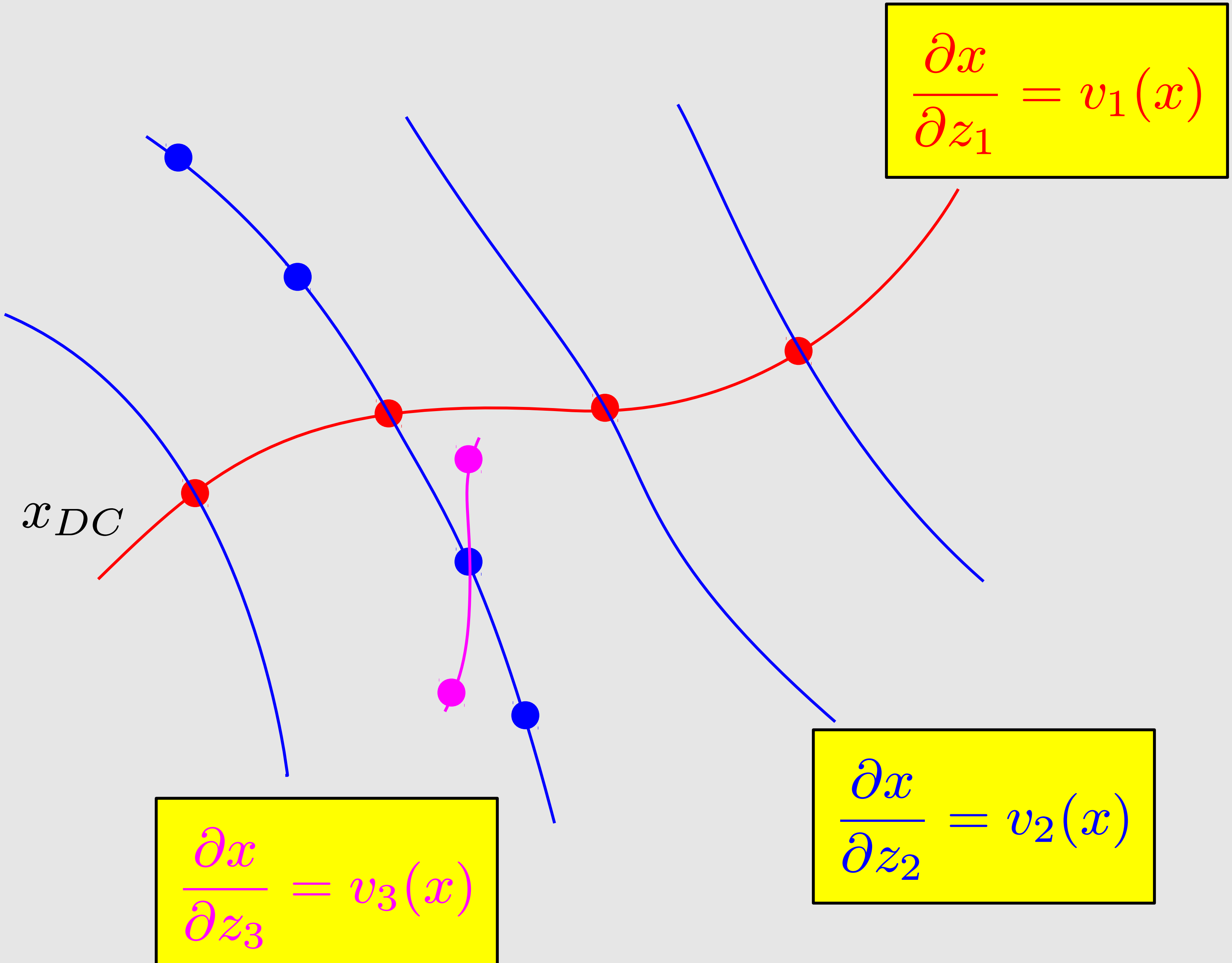
$V(x) = [v_1(x), \cdots, v_q(x)]$ is the projection matrix for the reduced linearized system (at x).

For example, Arnoldi algorithm generates a basis for

$$\mathcal{K}_q([G(x)]^{-1}, B) = \{[G(x)]^{-1}B, [G(x)]^{-2}B, \cdots, [G(x)]^{-q}B\}$$

However, this set of PDEs is over-determined.

Almost-Ideal Manifold Construction



Almost-Ideal Manifold Construction

Algorithm 1 Manifold Construction by Finding Integral Curves

```
Given the region to be parameterized (z_{i,min}, z_{i,max}), i ∈ [1, q];
Let x_0(0, ···, 0) = x_DC, where x_DC is the DC solution when u = 0;

X ← {x_0}, Z ← (0, ···, 0);
for i = 1 to q do
    for all x ∈ X do
        Integrate the integral curve equation
            ∂x/∂z_i = v_i(x)
        with initial condition x;
        X ← {x(z)}, Z ← z;
    end for
end for
Output X as the set of points on the manifold;
Output Z as the parameterization of the manifold for each point
x ∈ X.
```

Experimental Results

A Hand-Calculable Example

$$\frac{d}{dt}x_1 = -x_1 + x_2 - u(t)$$
$$\frac{d}{dt}x_2 = x_1^2 - x_2$$

$$f(x) = \begin{bmatrix} -x_1 + x_2 \\ x_1^2 - x_2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$G(x) = \begin{bmatrix} -1 & 1 \\ 2x_1 & -1 \end{bmatrix}, \quad [G(x)]^{-1} = \frac{1}{2x_1 - 1} \begin{bmatrix} 1 & 1 \\ 2x_1 & 1 \end{bmatrix}$$

DC and AC Manifold

$$[G(x)]^{-1} = \frac{1}{2x_1 - 1}\left[\begin{array}{cc} 1 & 1 \\ 2x_1 & 1 \end{array}\right], \quad B = \left[\begin{array}{c} -1 \\ 0 \end{array}\right]$$

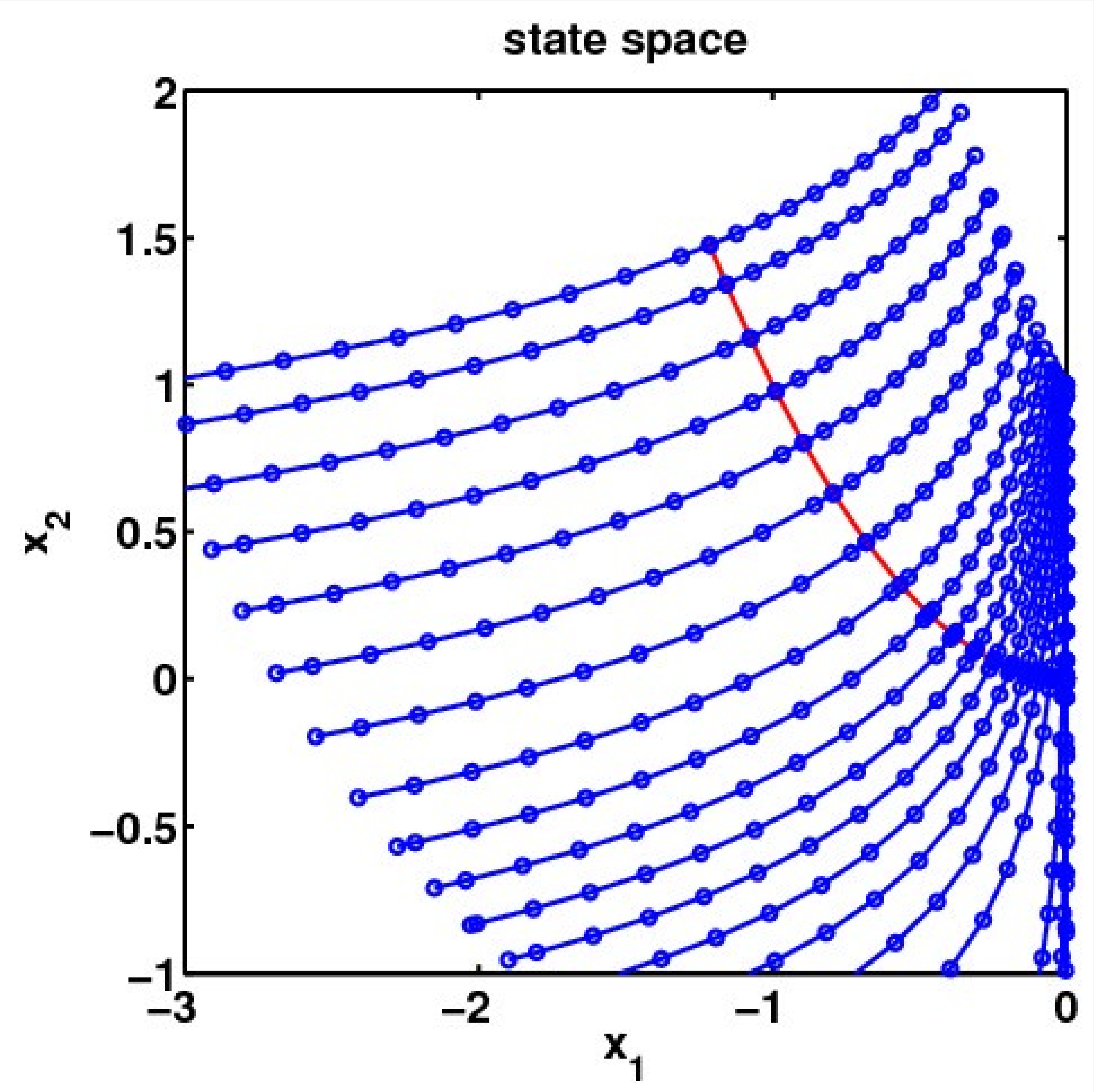
$$w_1(x) = [G(x)]^{-1}B = \frac{1}{2x_1 - 1}\left[\begin{array}{c} 1 \\ 2x_1 \end{array}\right]$$

$$w_2(x) = [G(x)]^{-2}B = \frac{1}{(2x_1 - 1)^2}\left[\begin{array}{c} -1 - 2x_1 \\ -4x_1 \end{array}\right]$$

DC manifold: $$\frac{\partial x}{\partial z_1} = v_1(x) = \frac{w_1(x)}{||w_1(x)||_2}$$

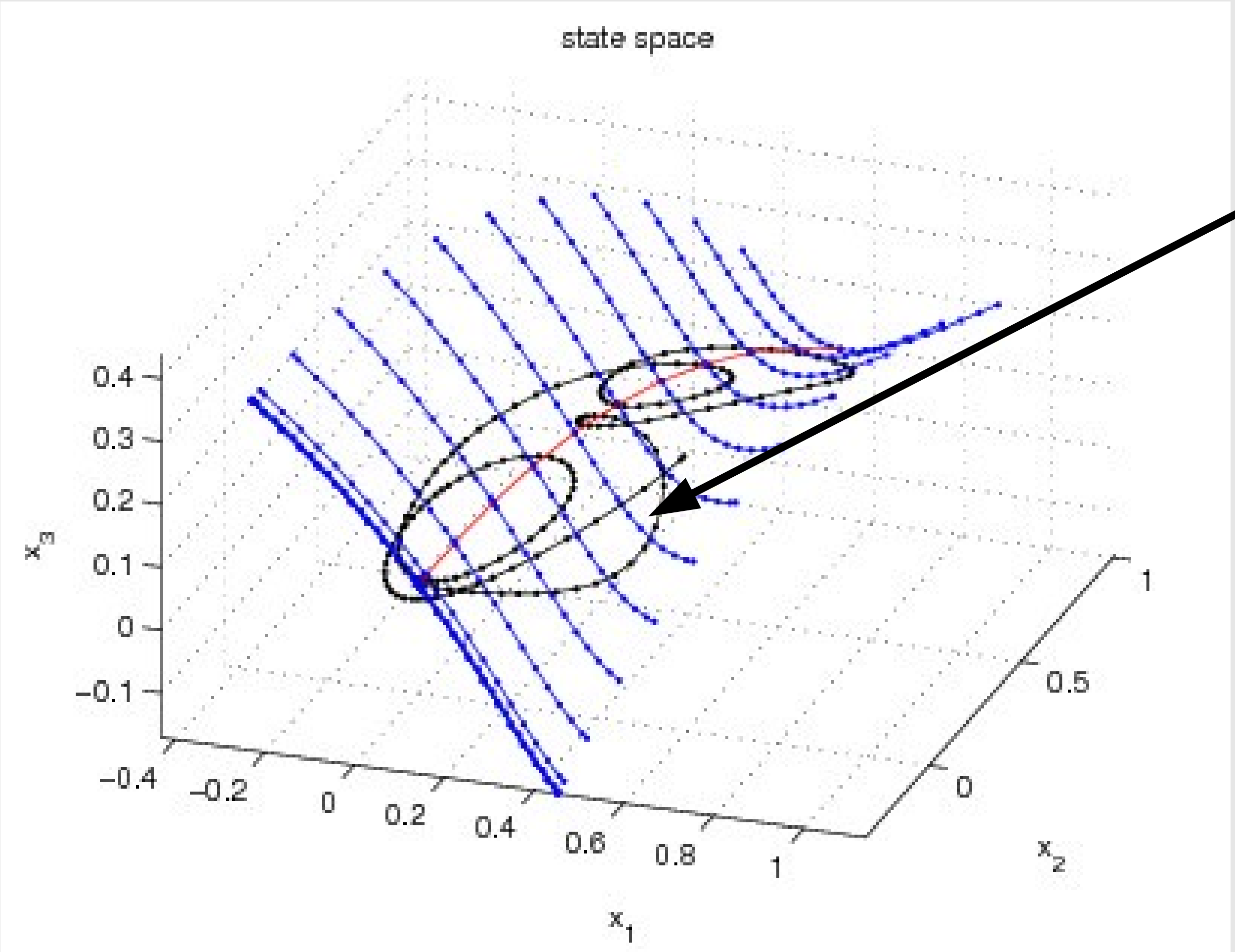
AC manifold: $$\frac{\partial x}{\partial z_1} = v_2(x) = \frac{w_2 - < w_2, v_1 > v_1}{||w_2 - < w_2, v_1 > v_1||_2}$$

DC and AC Manifold



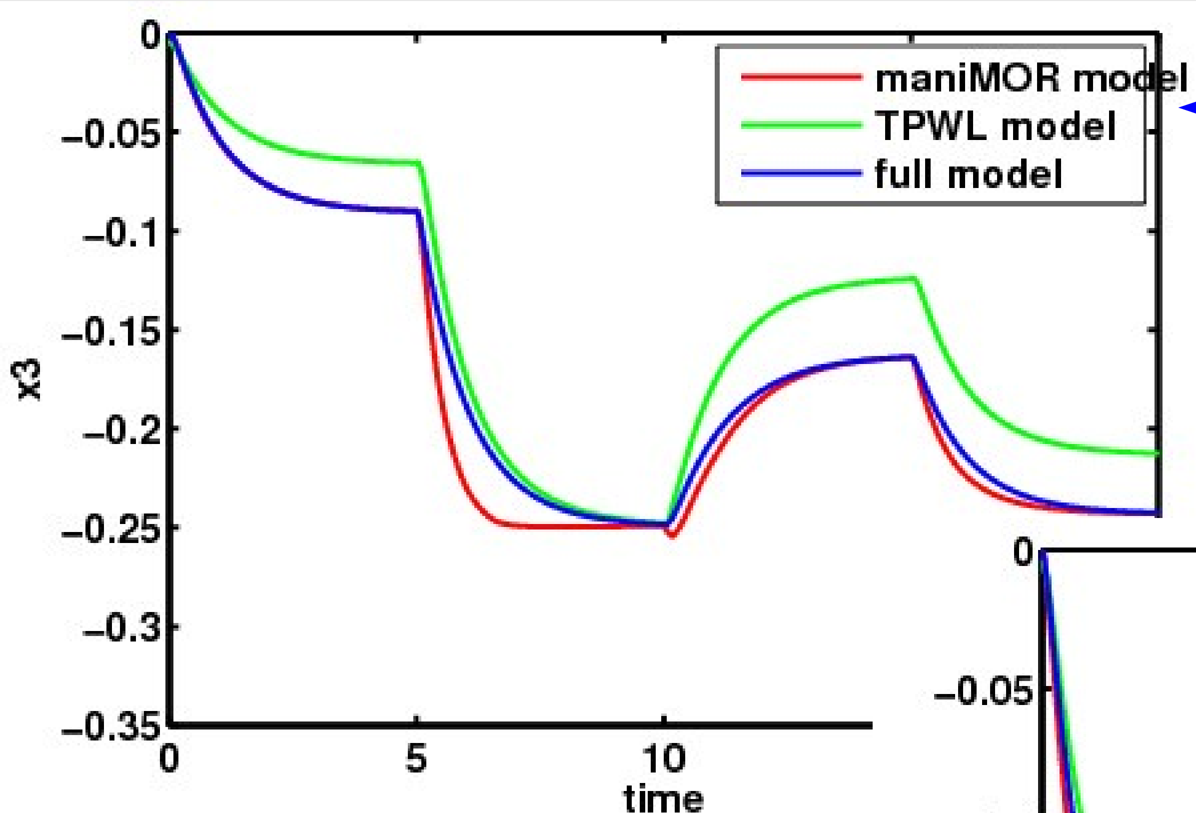
Application to MOR

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ x_1^2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$



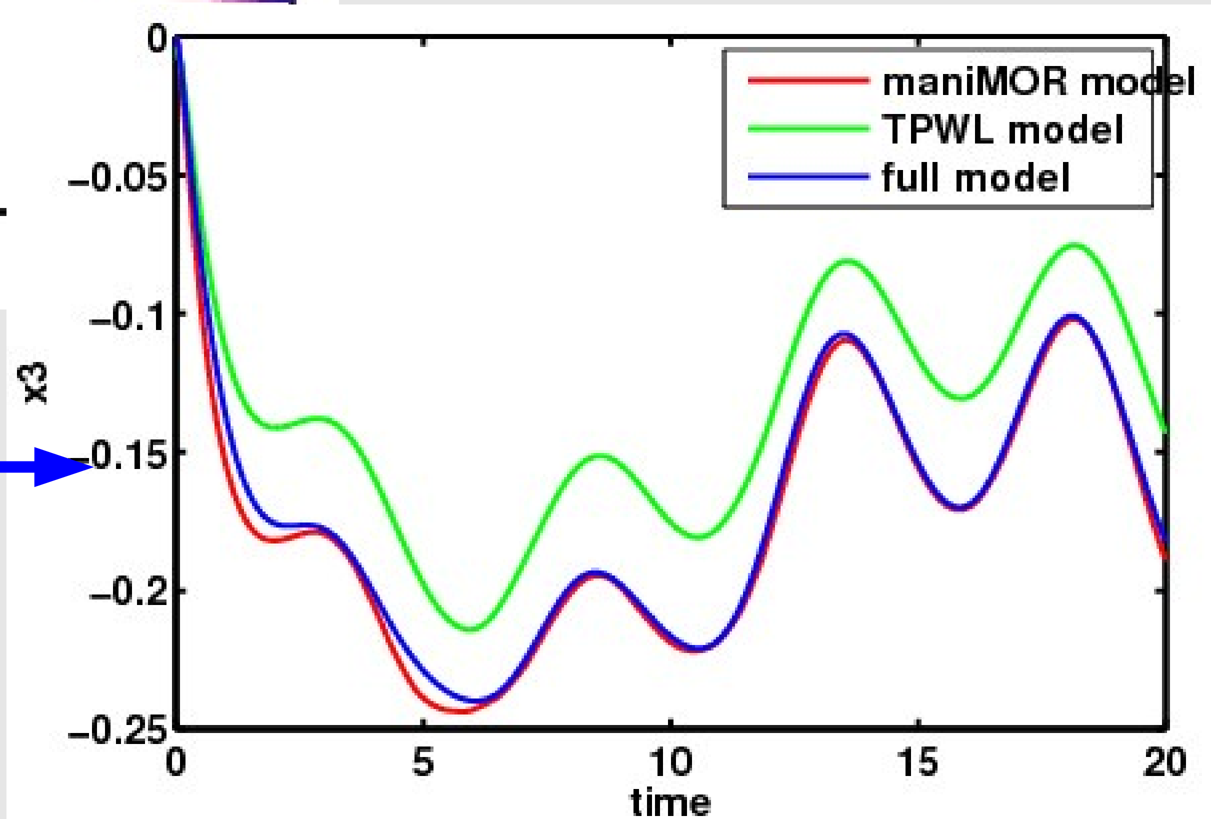
Trajectory of the full system stays close to the manifold

Simulation of the Reduced Order Model



Response to a step input

Response to a sinusoidal input



Conclusion

- **<u>Presented a manifold construction and parameterization procedure</u>**
  - Based on computing integral curves
  - Preserves local distance
  - Captures important system responses
    - Such as DC and AC responses
- **<u>Application to manifold-based MOR</u>**
  - Validated against several examples