Efficient Multi-tone Distortion Analysis of Analog Integrated Circuits

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***Abstract*— This paper introduces a novel approach to analyze distortion behavior in analog integrated circuits using a nonlinear frequency-domain method. This approach circumvents the difficulties and inaccuracies associated with device modeling required for the traditional Volterra series method and can handle circuits operating in more strongly nonlinear regimes. The efficiency of the method renders the analysis of large analog blocks practical. We present examples of multi-tone distortion analyses of industrial amplifiers and continuous time filters. The trend towards higher levels of integration, particularly for wireless applications, renders this method especially useful.**

I. Introduction

Designers of communications circuitry - both at baseband and radio frequencies - are concerned with the following questions which all come under the heading of "mildly non-linear" phenomena:

1. Harmonic distortion - drive the circuit under consideration with a sinusoidal input at a frequence ω and observe the amplitudes of the output harmonics at frequencies $k\omega$ $(k > 1)$
2. Intermodulation distortion - drive the circuit with *two* sinusoidal inputs (also called "tones") and observe the amplitude of the *intermodulation terms*;
3. Compression point - find that input power at which the ratio of a change in output power to a change in input power starts to drop less than one.

None of these effects are predicted by a linear small-signal model, however in each case the deviation from linearity is rather small.

The method of *Harmonic Balance* [1], [2], [3] is well-established as a simulation technique for nonlinear circuits driven by one or more periodic inputs. A particular advantage of the harmonic balance method is the ease with which it handles the "two tone" case of intermodulation distortion. The gist of the method is to write each waveform in the circuit as a Fourier series truncated to N coefficients, then replace the circuit's differential equations by a system of non-linear, algebraic equations involving the Fourier coefficients. This is possible because the derivative with respect to time of a Fourier series is just an algebraic operation. A numerical technique, such as Newton's method, is then employed to solve the resulting system of non-linear equations.

However, the system of equations can become rather large, since *each* waveform is replaced by a vector of N complex coefficients; N can be on the order of 128 for a two-tone study. A circuit with, say 1000 waveforms, would then generate a total system of 128000 equations in 128000 unknowns. Most numerical methods for solving larger systems of equations require the *Jacobian Matrix* of the equations to be formed at each step of an iterative process. In the case of harmonic balance equations, this matrix is rather dense, therefore, forming and factoring this matrix becomes a computational bottleneck for even medium-sized circuits (say, 50 transistors).

In this paper, we describe an implementation of a harmonic balance tool which circumvents the computational bottleneck associated with this large Jacobian matrix by using an *iterative linear solver* (such as the QMR algorithm [4]) which only requires multiplication of a vector by the Jacobian matrix or its transpose. We show that the time to multiply the Jacobian matrix with a vector grows only slightly faster than linearly with the total number of unknowns. A pre-conditioner for the iterative linear solver is proposed which facilitates rapid convergence of the iterative method, provided that the circuit is operating in the mildly non-linear regime as mentioned above. Our implementation enables the simulation of circuits much larger than can be handled with other implementations of harmonic balance. However, strongly non-linear behavior cannot be analyzed with the current tool.

II. Formulation of circuit equations

We assume that the circuit's behavior can be described by a system of equations of the form [5]

$$\mathbf{f}(\mathbf{x}, t) + \frac{d}{dt}\,\mathbf{q}(\mathbf{x}, t) = 0 \tag{1}$$

where:

- $\mathbf{x}(t)$ is an n-vector of circuit waveforms (in general, a mixture of voltage, currents, charges, and fluxes);
- $\mathbf{f}$ is an n-valued function of contributions from non-reactive elements to the the circuit equations;
- $\mathbf{q}$ is an n-valued function of reactive element contributions (charge for capacitors and flux for inductors).

Here, n is the dimension of the circuit's system of equations. Note that n depends to some extent on exactly how the circuit equations are formulated, but is always some indication of the "size" of the circuit under analysis.

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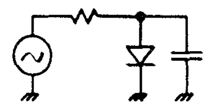

Fig. 1. Simple circuit example

For example Figure 1 shows a parallel combination of a diode and non-linear capacitor driven by a sinusoidal source with non-zero Thevenin output resistance. Let $v(t)$ be the voltage waveform across the diode and capacitor, and let $q(t)$ be the charge waveform stored on the capacitor. By current summation:

$$(v(t) - A\cos(\omega t))/R + i_D(v(t)) + \frac{d}{dt}q(t) = 0.$$

where i_D denotes the current through the diode. In addition, there is a non-linear, algebraic equation connecting the capacitors' charge with its applied voltage: $h(q(t), v(t)) = 0$. Thus, in the notation of (1), both $\mathbf{f}$ and $\mathbf{q}$ are mappings from R^2 into R^2:

$$\mathbf{f} : \begin{pmatrix} v(t) \\ q(t) \end{pmatrix} \mapsto \begin{pmatrix} \frac{(v(t) - A\cos(\omega t))}{R} + i_D(v(t)) \\ h(q(t), v(t)) \end{pmatrix}$$

and

$$\mathbf{q} : \begin{pmatrix} v(t) \\ q(t) \end{pmatrix} \mapsto \begin{pmatrix} q(t) \\ 0 \end{pmatrix}.$$

We are searching for a *steady-state* solution for Equation (1). One established method for finding such a solution is to assume that each steady-state waveform in the circuit can be represented by a truncated Fourier series

$$x^i(t) = \sum_{j=0}^{N-1} X^i_j w_j(t) = \mathbf{X}^{i^T}\mathbf{w}(t), \quad i = 1, 2, \ldots, n \qquad (2)$$

where:

- $\mathbf{w}(t)$ are the first N real Fourier basis functions, e.g.,

  $[1, +2\cos\omega t, -2\sin\omega t, +2\cos 2\omega t, -2\sin 2\omega t, \ldots]^T$

- $\mathbf{X}^i$ is a vector of Fourier coefficients for the i-th circuit variable.

For the case of a multi-tone analysis, the basis will contain functions for all frequencies of interest, their harmonics, and intermodulation products. The proper choice of the bases is studied extensively in the literature [6], [7] and is beyond the scope of this paper. For the sake of clarity we will base our derivations in the sequel on the one-tone Fourier basis, however, the results can be immediately generalized for multi-tone bases.

For a one-tone study, N might be on the order of 32, while for a two-tone study it might go as high as 256. The circuit determines n, which could be several thousand for a large network, especially if a complex transistor model is used. Substituting in (1), and expanding the derivative of $\mathbf{q}$ with respect to time we obtain:

$$\mathbf{f}(\mathbf{X}^T\mathbf{w}(t), t) + \left.\frac{\partial \mathbf{q}}{\partial \mathbf{x}}\right|_t \cdot \mathbf{X}^T\dot{\mathbf{w}}(t) + \frac{\partial \mathbf{q}}{\partial t} = 0 \qquad (3)$$

where

- $\mathbf{X} = [\mathbf{X}^1, \mathbf{X}^2, \ldots, \mathbf{X}^n]$ is the $N \times n$ matrix of circuit variable Fourier coefficients
- $\frac{\partial \mathbf{q}}{\partial t}$ is the n-vector of explicit derivatives w.r.t. time (null in most practical cases).

The steady-state solution waveforms to equation (3) are completely determined by the values of the $N \times n$ Fourier coefficients stored in matrix $\mathbf{X}$. We must now write a system of $N \times n$ algebraic equations equations from which we can determine the coefficients. We apply the standard method of ***point collocation*** [8] to system (3). That is, the equations are to hold at N different time points $t_1, t_2, \ldots, t_N$.

$$\begin{aligned} \mathbf{f}(\mathbf{X}^T\mathbf{w}(t_1), t_1) + \left.\frac{\partial \mathbf{q}}{\partial \mathbf{x}}\right|_{t_1} \cdot \mathbf{X}^T\dot{\mathbf{w}}(t_1) + \left.\frac{\partial \mathbf{q}}{\partial t}\right|_{t_1} &= 0 \\ &\vdots \\ \mathbf{f}(\mathbf{X}^T\mathbf{w}(t_N), t_N) + \left.\frac{\partial \mathbf{q}}{\partial \mathbf{x}}\right|_{t_N} \cdot \mathbf{X}^T\dot{\mathbf{w}}(t_N) + \left.\frac{\partial \mathbf{q}}{\partial t}\right|_{t_N} &= 0 \end{aligned} \qquad (4)$$

or, as a system of $n \times N$ equations in $n \times N$ unknowns

$$\mathbf{H}(\mathbf{X}) = 0. \qquad (5)$$

III. Solving the nonlinear system of equations

The system of algebraic nonlinear equations (5) is best solved by Newton-Raphson based algorithms [9], due to their quadratic convergence. However, Newton-Raphson based algorithms require computation of derivatives of $\mathbf{H}(\mathbf{X})$ with respect to $\mathbf{X}$ at various values of $\mathbf{X}$. The matrix of these derivatives - the so-called *Jacobian matrix* - is, however, large and rather dense. Existing harmonic balance programs are limited, in fact, by the CPU time and memory required to manipulate this matrix. In contrast, our implementation does not actually form this matrix in memory; instead it decomposes it into a sequence of simple linear transformations, that can be applied sequentially. It can be shown that the decomposition of the Jacobian matrix of $\mathbf{H}(\mathbf{X})$ takes the following form

$$\mathbf{J} = \frac{\partial \mathbf{H}}{\partial \mathbf{X}} = \mathbf{G}\mathbf{P}\Gamma\mathbf{P}^{-1} + \mathbf{C}\mathbf{P}\Gamma\mathbf{D}\mathbf{P}^{-1}. \qquad (6)$$

Here $\mathbf{G} = diag\{\mathbf{G}_1, \ldots, \mathbf{G}_N\}$ and $\mathbf{C} = diag\{\mathbf{C}_1, \ldots, \mathbf{C}_N\}$ are block diagonal matrices, with the blocks, $\mathbf{G}_i$ and $\mathbf{C}_i$, representing, respectively, derivatives of the two terms of the circuit equations (1) at the $i = 1, \ldots, N$ time points. These blocks are similar to the matrices used in standard time-domain analysis, hence are very sparse. Furthermore, Γ is a linear operator representing n applications of a time-to-frequency transformation, each of size N. For single-tone analysis the transformation is just the FFT; multi-tone analysis requires more involved, but computationally as efficient transforms[3]. Finally, $\mathbf{D}$, represents the time differentiation operator (a tridiagonal matrix) and $\mathbf{P}$ just a data permutation operator.

Every iteration of a Newton-Raphson type algorithm applied to the system (5) will typically require solving the following $nN \times nN$ linear system

$$\mathbf{J}\mathbf{z} = \mathbf{b} \tag{7}$$

The system matrix $\mathbf{J}$ can be quite large (say, $nN \geq$ 100,000) and dense, and thus expensive to factor. Therefore we consider iterative linear algebra methods [4] which solve (7) without factoring $\mathbf{J}$; instead they require repeated multiplications of $\mathbf{J}$ and maybe $\mathbf{J}^T$ to different vectors. By exploiting the special structure, (6), of the $\mathbf{J}$ the operations required for the multiplications are:

1. Multiplications of the block-diagonal matrices $\mathbf{G}$ and $\mathbf{C}$ with the vector. Due to the sparsity of typical circuit matrices this is accomplished in time $O(nN)$.
2. Application of the time-to-frequency transforms Γ, accomplished in time $O(nNlogN)$ when fast transforms are employed.
3. The applications of the differentiation operator $\mathbf{D}$, and the permutation operators $\mathbf{P}$ and $\mathbf{P}^{-1}$, which can both be done in time $O(nN)$.

The total computation time for the entire operation above is $O(nNlogN)$. This is considerably less expensive than the factorization of matrix $\mathbf{J}$.

Unfortunately, iterative linear algebra methods are not guaranteed to converge. The convergence is made much more robust by the use of a *preconditioner* [4]. That is, we apply the iterative method to a modified system

$$\tilde{\mathbf{J}}^{-1}\mathbf{J}x = \tilde{\mathbf{J}}^{-1}b \tag{8}$$

which has the same solution as (7). Intuitively, a good choice for $\tilde{\mathbf{J}}$ is a good approximation of $\mathbf{J}$ which is also relatively easy to invert. For a circuit that is mildly nonlinear, its linearization around the DC operating point represents such an approximation and, therefore, its Jacobian matrix suggest itself as an effective preconditioner. The application of this preconditioner (solving $\tilde{\mathbf{J}}z = b$) involves just N AC analyses of the linearized circuits at the N frequencies of interest.

IV. Examples

The first example is a 30 transistor MOS baseband amplifier from [10] analyzed in a closed-loop configuration. The configuration uses MOS transistors as tunable input resistors in order to adjust the gain. The designer was interested in the distortion contribution from the nonlinear input devices compared against ideal (linear) resistors.

First, as a calibration run, we performed a one-tone harmonic distortion study on the amplifier, and compared the results to a Fourier analysis of a transient run using a SPICE-class simulator. Of course, the transient simulation was allowed to go long enough to eliminate any transient effects. Highly accurate agreement (five significant digits) was obtained for the amplitudes of the harmonics as predicted by the two simulation methods.

Then, a two-tone study, with excitations at 1kHz and 1.01kHz, was performed using the harmonic balance implementation. Figures 2 and 3 show the results of the intermodulation study using ideal resistors versus biased

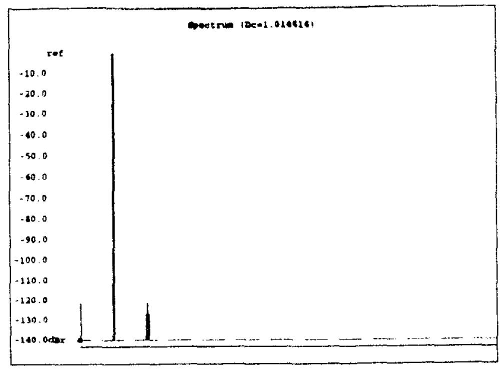

Fig. 2. Amplifier output spectrum with ideal gain setting devices

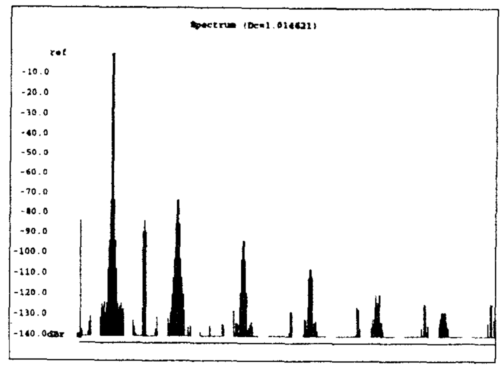

Fig. 3. Amplifier output spectrum with MOS gain setting devices

transistors. The significant distortion contribution from the nonlinearity of the input transistors is clearly seen. Absolutely robust convergence was observed on this example. In the notation of the paper, n was 30 and N was 128, given a final matrix dimension of 3800. A maximum of 6 circuit Jacobian evaluations were needed for each linear solve during the 8 iterations of Newton's method needed to converge to the final answer. This simulation was accomplished in less than three minutes. The harmonic balance analysis of this circuit is fast, accurate, and independent of the frequency separation of the input tones. By comparison, the use of the transient method, mentioned above, to perform this two-tone study would be tedious and problematic. First, because of the small separation of the two tones a long transient run is needed to capture the steady state. Moreover, the postprocessing of the transient data severely limits the dynamic range of the analysis.

Our second example is a 380 transistors low-distortion amplifier, also implemented in a MOS technology. We performed a one-tone harmonic distortion analysis. This circuit had 216 waveforms, and was analyzed with 27 Fourier coefficients, for a total system size of 5832. For an input amplitude of 300 mV, the convergence of the program is fast and robust – only five Newton iterations and a worst-case iteration count of 58 for the linear solver. Total simulation time is about five minutes on a fast scientific workstation. As predicted by the designer the spectrum of the

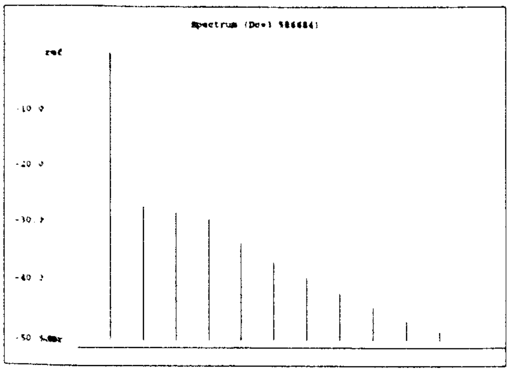

Fig. 4. Amplifier signal spectrum with 300 mV stimulus

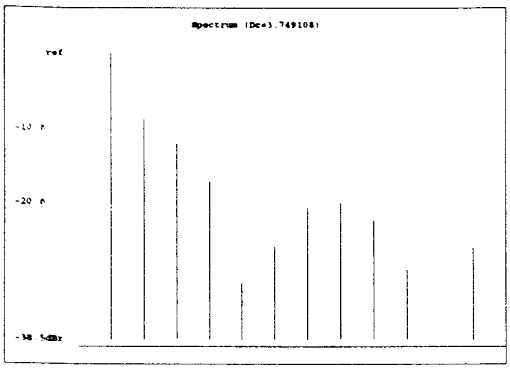

Fig. 5. Amplifier signal spectrum with 375 mV stimulus

output signal was extremely pure with the largest harmonic more than 80 dB below the fundamental. However not all waveforms in the circuit were as pure. Figure 4 shows the voltage spectrum of the most distorted waveform. Even though this waveform indicates that the circuit has significant nonlinear behavior, our method converges robustly. However, as the input amplitude was increased above 400 mV, the method began to slow down, then failed to converge. The reason is that the degree of nonlinearity depends on the input amplitude and our preconditioner stops being an adequate approximation to the system Jacobian when the circuit becomes highly nonlinear. Figure 5 shows the spectrum of the same signal as Figure 4 but with an input amplitude of 375 mV, which represents about the limit of our technique. This waveform is quite distorted as confirmed by the time-domain plot of Figure 6.

V. Conclusion

This paper describes an algorithm for the efficient computation of the steady-state response of mildly nonlinear circuits subjected to one or more sinusoidal excitations. We formulate the circuit equations as a harmonic-balance problem, and we significantly accelerate the solution time by employing iterative methods for solving linear systems. Our method is based on two novel results. The first, is the decomposition of the large and dense Jacobian matrix of the harmonic balance system of equations into a number

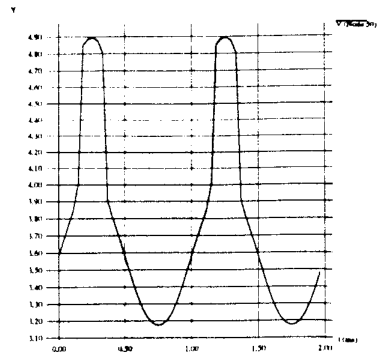

Fig. 6. Amplifier time-domain signal with 375 mV stimulus

of relatively inexpensive-to-apply linear operators. This decomposition makes possible the solution of very large problems through the use of iterative linear solvers which avoid the storage and factorization of the Jacobian matrix. However, iterative linear solvers converge reliably only if a good preconditioner is available. The second contribution of this paper is the use of the linearized circuit Jacobian matrix as preconditioner. This preconditioner is based on solid engineering intuition, is adequate for all circuits operating in a mildly nonlinear regime, and is inexpensive to apply.

VI. Acknowledgements

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