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Phase Noise in Oscillators: A Unifying Theory and Numerical Methods for Characterisation

Alper Demir Amit Mehrotra* Jaijeet Roychowdhury
{alpdemir, mehrotra, jaijeet}@research.bell-labs.com

Bell Laboratories
Murray Hill, New Jersey, USA

Abstract

Phase noise is a topic of theoretical and practical interest in electronic circuits, as well as in other fields such as optics. Although progress has been made in understanding the phenomenon, there still remain significant gaps, both in its fundamental theory and in numerical techniques for its characterisation. In this paper, we develop a solid foundation for phase noise that is valid for any oscillator, regardless of operating mechanism. We establish novel results about the dynamics of stable nonlinear oscillators in the presence of perturbations, both deterministic and random. We obtain an exact, nonlinear equation for phase error, which we solve without approximations for random perturbations. This leads us to a precise characterisation of timing jitter and spectral dispersion, for computing which we develop efficient numerical methods. We demonstrate our techniques on practical electrical oscillators, and obtain good matches with measurements even at frequencies close to the carrier, where previous techniques break down.

1 Introduction

Oscillators are ubiquitous in physical systems, especially electronic and optical ones. For example, in radio frequency (RF) communication systems, they are used for frequency translation of information signals and for channel selection. Oscillators are also present in digital electronic systems which require a time reference, i.e., a clock signal, in order to synchronise operations.

Noise is of major concern in oscillators, because introducing even small noise into an oscillator leads to dramatic changes in its frequency spectrum and timing properties. This phenomenon, peculiar to oscillators, is known as *phase noise* or *timing jitter*. A perfect oscillator would have localized tones at discrete frequencies (i.e., harmonics), but any corrupting noise spreads these perfect tones, resulting in high power levels at neigbouring frequencies. This effect is the major contributor to undesired phenomena such as interchannel interference, leading to increased bit-error-rates (BER) in RF communication systems. Another manifestation of the same phenomenon, jitter, is important in clocked and sampled-data systems: uncertainties in switching instants caused by noise lead to synchronisation problems. Characterising how noise affects oscillators is therefore crucial for practical applications. The problem is challenging, since oscillators constitute a special class among noisy physical systems: their *autonomous* nature makes them unique in their response to perturbations.

Considerable effort has been expended over the years in understanding phase noise and in developing analytical, computational and experimental techniques for its characterisation (see Section 3 for a brief review). Despite the importance of the problem and the large number of publications on the subject, a consistent and general treatment, and computational techniques based on a sound theory, appear to be still lacking. In this work, we provide a novel, rigorous theory for phase noise and derive efficient numerical methods for its characterisation. Our techniques and results are general; they are applicable to *any* oscillatory system, electrical (resonant, ring, relaxation, etc.) or otherwise (gravitational, optical, mechanical, biological, etc.). The main ideas behind our approach, and our contributions, are outlined in Section 2. We apply our numerical techniques to a variety of practical oscillator designs and obtain good matches against measurements.

The paper is organised as follows. In Section 2, we present some preliminaries and an overview of the main results of the paper, and in Section 3, we give a brief review of the previous work. In Section 4, we consider the traditional approach (linearisation) to analysing perturbed nonlinear systems, and show how this procedure is not consistent for autonomous oscillators. In Section 5, we derive a nonlinear equation that exactly captures how perturbations result in phase noise. In Section 6, we solve this equation with random perturbations and arrive at a stochastic description of phase deviation, from which we derive timing jitter. Next, in Section 7, we use this stochastic characterisation to calculate the correct shape of the oscillator's spectrum with phase noise. In Section 8, we derive several quantities commonly used in oscillator design to quantify jitter and spectral properties. In Section 9, we address the problem of computing these quantities efficiently and develop numerical methods that can easily be implemented in existing simulators. Finally, in Section 10, we apply our methods to practical electrical oscillators. All proofs and discussion of mathematical background are omitted due to space limitations.

*visiting from the University of California, Berkeley.

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2 Preliminaries and overview

The dynamics of any autonomous system without undesired perturbations can be described by a system of differential equations:[1]

$$\dot{x} = f(x) \tag{1}$$

where $x \in \mathbb{R}^n$ and $f(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$. We assume that $f(\cdot)$ satisfies the conditions of the Picard-Lindelöf existence and uniqueness theorem for initial value problems [2]. We consider systems that have an asymptotically orbitally stable[2] periodic solution $x_s(t)$ (with period T) to (1), i.e., a stable limit cycle in the n-dimensional solution space. We are interested in the response of such systems to a small state-dependent perturbation of the form $B(x)b(t)$ where $B(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^{n\times p}$ and $b(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^p$. Hence the perturbed system is described by

$$\dot{x} = f(x) + B(x)b(t) \tag{2}$$

Let the exact solution of the perturbed system in (2) be $z(t)$.

Although our eventual intent is to understand the response of the oscillator when $b(t)$ is random noise, it is useful to consider first the case when $b(t)$ is a known deterministic signal. We carry out a rigorous analysis of this case in Section 5 and obtain the following results:

1. the unperturbed oscillator's periodic response $x_s(t)$ is modified to $x_s(t+\alpha(t)) + y(t)$ by the perturbation, where:
   (a) $\alpha(t)$ is a changing time shift, or *phase deviation*, in the periodic output of the unperturbed oscillator.
   (b) $y(t)$ is an additive component, which we term the *orbital deviation*, to the phase-shifted oscillator waveform.
2. $\alpha(t)$ and $y(t)$ can always be chosen such that:
   (a) $\alpha(t)$ will, in general, keep increasing with time even if the perturbation $b(t)$ is always small.
   (b) the orbital deviation $y(t)$, on the other hand, will always remain small.

[1] For notational simplicity, we use the ODE formulation throughout the paper to describe the dynamics of an autonomous system. The results and the numerical methods we present can be extended [1] for the MNA (Modified Nodal Analysis) formulation (i.e., DAE formulation) given by $d/dt\, q(x) + f(x) = 0$.

[2] After any small disturbance that does not persist, the system asymptotically settles back to the original limit cycle. See [2] for a precise definition of this stability notion.

These results concretise existing intuition amongst designers about oscillator operation. Our proof of these facts is mathematically rigorous; further, we derive equations for $\alpha(t)$ and $y(t)$ which lead to *qualitatively different results* about phase noise compared to previous attempts. This is because our results are based on a new nonlinear perturbation analysis that is valid for oscillators, in contrast to previous approaches that rely on linearisation. We show in Section 4 that analysis based on linearisation is not consistent for oscillators and results in non-physical predictions.

Next, we consider the case where the perturbation $b(t)$ is random noise – this situation is important for determining practical figures of merit like zero-crossing jitter and spectral purity (i.e., spreading of the power spectrum)[3]. Jitter and spectral spreading are in fact closely related, and both are determined by the manner in which $\alpha(t)$, now also a random process, spreads with time. We consider random perturbations in detail in Sections 6 and 7, and establish that:

1. the average spread of the jitter (mean-square jitter) increases precisely *linearly* with time.
2. the power spectrum of the perturbed oscillator is a *Lorentzian*[4] about each harmonic.
3. a *single scalar constant c* is sufficient to describe jitter and spectral spreading in a noisy oscillator.
4. the oscillator's output is a *stationary* stochastic process.

These results have important implications. The Lorentzian shape of the spectrum implies that the power spectral density at the carrier frequency and its harmonics has a finite value, and that the total carrier power is preserved despite spectral spreading due to noise. Previous analyses based on linear time-invariant (LTI) or linear time-varying (LTV) concepts erroneously predict infinite noise power density at the carrier, as well as infinite total integrated power. That the oscillator output is stationary is surprising at first sight, since oscillators are nonlinear systems with periodic swings, hence it might be expected that output noise power would change periodically as in forced systems. However, it must be remembered that while forced systems are supplied with an external time reference (through the forcing), oscillators are not. Cyclostationarity in the oscillator's output would, by definition, imply a time reference. Hence the stationarity result reflects the fundamental fact that noisy autonomous systems cannot provide a perfect time reference.

3 Previous work

A great deal of literature is available on the phase noise problem. Here we mention only some selected works. Most investigations of electronic oscillators aim to provide insight into frequency-domain properties of phase noise, in order to develop rules for designing practical oscillators; well-known references include [4, 5, 6, 7, 8]. Usually, these approaches apply linear time-invariant (LTI) analysis to high-Q or quartz-crystal type oscillators designed using standard feedback topologies. Arguments based on deterministic perturbations are used to show that the spectrum of the oscillator response varies as $1/f^2$ times the spectrum of the perturbation. While often of great practical importance, such analyses often require large simplifications of the problem, and skirt fundamental issues such as why noisy oscillators exhibit spectral dispersion whereas forced systems do not.

Attempts to improve on LTI analysis have borrowed from linear time-varying (LTV) analysis methods for forced (nonoscillatory) systems (e.g., [9, 10, 11, 12]). LTV analyses can predict spectra more accurately than LTI ones in some frequency ranges; however, LTV techniques for forced systems retain nonphysical artifacts of LTI analysis (such as infinite output power) and provide no real insight into the basic mechanism generating phase noise.

Possibly the most general and rigorous treatment of phase noise to date has been that of Kärtner [13]. In this work, the oscillator response is decomposed into phase and magnitude components, and a differential equation is obtained for phase error. By solving a linear, small-time approximation to this equation with stochastic inputs, Kärtner obtains the correct Lorentzian spectrum for the power spectral density due to phase noise. Despite these advances, certain gaps remain, particularly with respect to the derivation and solution of the differential equation for phase error.

[3]The deterministic perturbation case is also of interest, for, e.g., phenomena such as mode locking in forced oscillators. We consider this case elsewhere [3].

[4]A Lorentzian is the shape of the squared magnitude of a one-pole lowpass filter transfer function.

Recently, Hajimiri [14] has proposed a phase noise analysis based on a conjecture for decomposing perturbations into two (orthogonal) components, generating purely phase and amplitude deviations respectively. While this intuition is similar to Kärtner's approach [13], other aspects of Hajimiri's treatment (e.g., stochastic characterisation for phase deviation and the spectrum calculation) are essentially equivalent to LTV analysis. Unfortunately, the conjecture for orthogonally decomposing the perturbation into components that generate phase and amplitude deviations, while intuitively appealing, can be shown to be invalid [15]. Design intuition resulting from the conjecture about noise source contributions can also be misleading.

In summary, the available literature often identifies basic and useful facets of phase noise separately, but lacks a rigorous unifying theory clarifying its fundamental mechanism. Furthermore, existing numerical methods for phase noise are based on forced-system concepts which are inappropriate for oscillators and can generate incorrect predictions.

4 Perturbation analysis using linearisation

The traditional approach to analysing perturbed nonlinear systems is to linearise about the unperturbed solution, under the assumption that the resultant deviation[5] will be small. Let this deviation be $w(t)$, i.e., $z(t) = x_s(t) + w(t)$. Substituting this expression for $z(t)$ in (2), replacing $f(x_s(t) + w(t))$ by its first order Taylor series expansion, and approximating $B(x)$ with $B(x_s)$ (assuming $w(t)$ "small"), we obtain

$$\dot{w} \approx \left.\frac{\partial f(x)}{\partial x}\right|_{x_s(t)} w(t) + B(x_s(t))b(t) = A(t)w(t) + B(x_s(t))b(t) \quad (3)$$

where the Jacobian $A(t) = \left.\frac{\partial f(x)}{\partial x}\right|_{x_s(t)}$ is T-periodic. Here, we used the fact that $x_s(t)$ satisfies (1). Now, we would like to solve for $w(t)$ in (3) to see if our assumption that it is small is indeed justified. For this, we use results from Floquet theory [2, 16] as follows[6].

The state transition matrix for the homogeneous part of (3) is given by

$$\Phi(t,s) = U(t)\exp(D(t-s))V(s) = \sum_{i=1}^{n} u_i(t)\exp(\mu_i(t-s))v_i^T(s) \quad (4)$$

where $U(t)$ is a T-periodic nonsingular matrix, $V(t) = U^{-1}(t)$ and $D = diag[\mu_1, \ldots, \mu_n]$, where μ_i are the *Floquet (characteristic) exponents.* $\exp(\mu_i T)$ are called the *characteristic multipliers.* $u_i(t)$ are the columns of $U(t)$ and $v_i^T(t)$ are the rows of $V(t) = U^{-1}(t)$.

Remark 4.1 *$\{u_1(t), u_2(t), \ldots, u_n(t)\}$ and $\{v_1(t), v_2(t), \ldots, v_n(t)\}$ both span $\mathbb{R}^n$ and satisfy the* biorthogonality conditions *$v_i^T(t)u_j(t) = \delta_{ij}$ for every t. Note that, in general, $U(t)$ itself is not an orthogonal matrix.*

Let us first consider the homogeneous part of (3), the solution of which is given by

$$w_H(t) = \sum_{i=1}^{n} u_i(t)\exp(\mu_i t)v_i^T(0)w(0) \quad (5)$$

where $w(0)$ is the initial condition. Next, we will show that one of the terms in the summation in (5) does not decay with t.

Lemma 4.1

- *The unperturbed oscillator (1) has a non-trivial T-periodic solution $x_s(t)$ if and only if the number 1 is a characteristic multiplier of the homogeneous part of (3), or equivalently, one of the Floquet exponents satisfies $\exp(\mu_i T) = 1$.*
- *The time-derivative of the periodic solution $x_s(t)$ of (1), i.e., $\dot{x}_s(t)$, is a solution of the homogeneous part of (3).*

[5]By deviation we refer to the difference between the solutions of the perturbed and unperturbed systems.

[6]The reader who is unfamiliar with Floquet theory is encouraged to review it before continuing.

Remark 4.2 *One can show that if 1 is a characteristic multiplier, and the remaining $n-1$ Floquet exponents satisfy $|\exp(\mu_i T)| < 1$, $i = 2,\ldots,n$, then the periodic solution $x_s(t)$ of (1) is asymptotically orbitally stable, and it has the asymptotic phase property [2].[7] Moreover, if any of the Floquet exponents satisfy $|\exp(\mu_i T)| > 1$, then the solution $x_s(t)$ is orbitally* unstable.

Without loss of generality, we choose $\mu_1 = 0$ and $u_1(t) = \dot{x}_s(t)$.

Remark 4.3 *With $u_1(t) = \dot{x}_s(t)$, we have $v_1^T(t)\dot{x}_s(t) = 1$ and $v_1^T(t)u_j(t) = 0$, $j = 2,\ldots,n$. $v_1(t)$ will play an important role in the rest of our treatment.*

Next, we obtain the particular solution of (3), given by

$$w_P(t) = \sum_{i=1}^{n} u_i(t) \int_0^t \exp(\mu_i(t-r)) v_i^T(r) B(x_s(r)) b(r) \mathrm{d}r \quad (6)$$

The first term in the above summation is given by $u_1(t)\int_0^t v_1^T(r)B(x_s(r))b(r)\mathrm{d}r$, since $\mu_1 = 0$. If the integrand has a nonzero average value, then the deviation $w(t)$ in (3) will grow unbounded. Hence, the assumption that $w(t)$ is small becomes invalid and the linearised perturbation analysis is inconsistent.

When the perturbation $b(t)$ is a vector of uncorrelated white noise sources, one can show that the variances of the entries of $w(t)$ can grow unbounded. Thus, the assumption that the deviation $w(t)$ stays small[8] is also invalid for the stochastic perturbation case.

5 Nonlinear perturbation analysis for phase deviation

As seen in the previous section, traditional perturbation techniques do not suffice for analysing oscillators. In this section, a novel nonlinear perturbation analysis suitable for oscillators is presented.

The new analysis proceeds along the following lines:

1. Rewrite (2) with the (small) perturbation $B(x)b(t)$ split into two small parts $b_1(x,t)$ and $\tilde{b}(x,t)$:
$$\dot{x} = f(x) + b_1(x,t) + \tilde{b}(x,t) \quad (7)$$
2. Choose the first perturbation term $b_1(x,t)$ in such a way that its effect is to create only *phase errors* to the unperturbed solution. In other words, show that the equation
$$\dot{x} = f(x) + b_1(x,t) \quad (8)$$
is solved by $x_p(t) = x_s(t+\alpha(t))$ for a certain function $\alpha(t)$, called the *phase deviation*. It will be seen that $\alpha(t)$ can grow unboundedly large with time even though the perturbation $b_1(x,t)$ remains small.
3. Now treat the remaining term $\tilde{b}(x,t)$ as a small perturbation to (8), and perform a consistent traditional perturbation analysis in which the resultant deviations from $x_p(t)$ remain small. I.e., show that $z(t) = x_s(t+\alpha(t)) + y(t)$ solves (7) for a certain $y(t)$ that *remains small* for all t. $y(t)$ will be called the *orbital deviation*.

We start by defining $\alpha(t)$ concretely through a differential equation.

Definition 5.1 *Define $\alpha(t)$ by*

$$\frac{d\alpha(t)}{dt} = v_1^T(t+\alpha(t))B(x_s(t+\alpha(t)))b(t), \quad \alpha(0) = 0 \quad (9)$$

Remark 5.1 *$\alpha(t)$ can grow unbounded even if $b(t)$ remains small. For example, consider the case where $b(t)$ is a small positive constant $\varepsilon \ll 1$, $B \equiv 1$, and $v_1(t)$ is a constant k. Then $\alpha(t) = k\varepsilon t$.*

Having defined $\alpha(t)$, we are in a position to split $B(x)b(t)$ into $b_1(x,t)$ and $\tilde{b}(x,t)$:

[7] Note that this is a sufficient condition for asymptotic orbital stability, not a necessary one. We assume that this sufficient condition is satisfied by the system and the periodic solution $x_s(t)$.

[8] The notion of "staying small" is quite different for a stochastic process than the one for a deterministic function. For instance, a Gaussian random variable can take arbitrarily large values with nonzero probability even when its variance is "small". We say that a stochastic process is "bounded" when its variance is bounded, even though some of its sample paths (representing a nonzero probability) can grow unbounded.

Definition 5.2 *Let*

$$b_1(x,t) = c_1(x,t)u_1(t+\alpha(t)), \quad \text{and} \quad (10)$$

$$\tilde{b}(x,t) = B(x)b(t) - b_1(x,t) = \sum_{i=2}^{n} c_i(x,t)u_i(t+\alpha(t)), \quad (11)$$

where the scalars $c_i(x,t) = v_i^T(t+\alpha(t))B(x)b(t)$

Note that $b_1(x,t)$ is obtained by projecting the original perturbation along the time-varying direction $u_1(t+\alpha(t))$. u_i, v_i are the Floquet vectors in Remark 4.1.

Lemma 5.1 *$x_p(t) = x_s(t+\alpha(t))$ solves (8).*

Lemma 5.1 states that the $b_1(x,t)$ component causes deviations only along the limit cycle, i.e., phase deviations. Next, we show that the remaining perturbation component $\tilde{b}(x,t)$ perturbs $x_p(t)$ only by a small amount $y(t)$, provided $b(t)$ is small.

Lemma 5.2 *For $b(t)$ sufficiently small, the mapping $t \mapsto t+\alpha(t)$ is invertible.*

Definition 5.3 *Let $b(t)$ be small enough that $\hat{t}(t) = t+\alpha(t)$ is invertible. Then define $\hat{b}(\cdot)$ by $\hat{b}(\hat{t}) = b(t)$, and $y(t)$ by*

$$y(t) = \sum_{i=2}^{n} u_i(\hat{t}) \int_0^{\hat{t}} \exp(\mu_i(\hat{t}-r)) v_i^T(r) B(x_s(r)) \hat{b}(r) \mathrm{d}r \quad (12)$$

where $\hat{t} = t+\alpha(t)$.

Remark 5.2 *Note that the index of the summation in (12) starts from 2. Since $|\exp(\mu_i T)| < 1, i \geq 2$ (due to asymptotic orbital stability), this implies that $y(t)$ is within a constant factor of $b(t)$, hence small.*

Theorem 5.1 *If $b(t)$ is small (implying that $y(t)$ in Definition 5.3 is also small), then $z(t) = x_p(t) + y(t)$ solves (7) to first order in $y(t)$.*

6 Stochastic characterisation of the phase deviation α

We now find the probabilistic characterisation of the phase deviation α (Definition 5.1) as a stochastic process when the perturbation $b(t)$ is a vector of uncorrelated[9] Gaussian white noise sources. We will treat (9) as a stochastic differential equation [17, 18].

We will follow the below procedure to find an adequate probabilistic characterisation of the phase deviation α for our purposes:

1. We first calculate the time-varying *probability density function* (PDF) $p_\alpha(\eta,t)$ of α defined as
$$p_\alpha(\eta,t) = \frac{\partial \mathrm{P}(\alpha(t) \leq \eta)}{\partial \eta} \quad t \geq 0$$
where $\mathrm{P}(.)$ denotes the *probability measure*, and show that it becomes the PDF of a Gaussian random variable asymptotically with t. A Gaussian PDF is completely characterised by the mean and the variance of the random variable. We show that $\alpha(t)$ becomes, asymptotically with time, a Gaussian random variable with a constant (as a function of t) mean and a variance that is linearly increasing with time.[10]
2. The time-varying PDF $p_\alpha(\eta,t)$ does not provide any correlation information between $\alpha(t)$ and $\alpha(t+\tau)$ that is needed for the evaluation of its spectral characteristics. We then calculate this correlation to be
$$\mathrm{E}[\alpha(t)\alpha(t+\tau)] = m^2 + c\min(t,t+\tau)$$
where m and c are scalar constants.
3. We then show that $\alpha(t_1)$ and $\alpha(t_2)$ become *jointly* Gaussian asymptotically with time, which does not follow immediately from the fact that they are individually Gaussian.

[9] The extension to correlated noise sources is trivial. We consider uncorrelated noise sources for notational simplicity. Moreover, various noise sources in electronic devices usually have independent physical origin, and hence they are modeled as uncorrelated stochastic processes.

[10] The fact that $\alpha(t)$ is a Gaussian random variable for every t does not imply that α is a Gaussian stochastic process. Individually Gaussian random variables are not necessarily jointly Gaussian.

Starting with the stochastic differential equation (9) for α, one can derive a partial differential equation, known as the *Fokker-Planck equation* [18, 19], for the time-varying PDF $p_\alpha(\eta,t)$. The Fokker-Planck equation for $\alpha(t)$ takes the form

$$\begin{aligned}\frac{\partial p_\alpha(\eta,t)}{\partial t} &= -\frac{\partial}{\partial \eta}\left(\lambda p_\alpha(\eta,t)\frac{\partial v^T(t+\eta)}{\partial \eta}v(t+\eta)\right) \\ &+ \frac{1}{2}\frac{\partial^2}{\partial \eta^2}\left(v^T(t+\eta)v(t+\eta)p_\alpha(\eta,t)\right)\end{aligned} \tag{13}$$

where $v^T(t) = v_1^T(t)B(x_s(t))$, and $0 \le \lambda \le 1$ depends on the definition of the stochastic integral [18] used to interpret the stochastic differential equation in (9). We would like to solve (13) for $p_\alpha(\eta,t)$. It turns out that $p_\alpha(\eta,t)$ becomes a *Gaussian* PDF asymptotically with *linearly* increasing variance. We show this by first solving for the *characteristic function* $F(\omega,t)$ of $\alpha(t)$, which is defined by

$$F(\omega,t) = \mathrm{E}\left[\exp\left(j\omega\alpha(t)\right)\right] = \int_{-\infty}^{\infty}\exp\left(j\omega\eta\right)p_\alpha(\eta,t)\mathrm{d}\eta$$

Since both $v_1^T(.)$ and $B(x_s(.))$ are T-periodic in their arguments, $v^T(.)$ is also periodic in its argument with period T. Hence we can expand $v^T(t)$ into its Fourier series: $v^T(t) = \sum_{i=-\infty}^{\infty} V_i^T \exp(ji\omega_0 t)$ where $\omega_0 = 2\pi/T$.

Lemma 6.1 *The characteristic function of* $\alpha(t)$, $F(\omega,t)$, *satisfies*

$$\begin{aligned}\frac{\partial F(\omega,t)}{\partial t} &= \sum_{i=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} V_i^T V_k^* \exp(j\omega_0(i-k)t) \\ &\left(-\lambda\omega_0 i\omega - \frac{1}{2}\omega^2\right)F(\omega_0(i-k)+\omega,t)\end{aligned} \tag{14}$$

where $*$ *denotes complex conjugation.*

Theorem 6.1 *(14) has a solution that becomes the characteristic function of a Gaussian random variable asymptotically with time:*

$$\lim_{t\to\infty} F(\omega,t) = \exp\left(j\omega\mu(t) - \frac{\omega^2\sigma^2(t)}{2}\right) \tag{15}$$

solves (14), where $\mu(t) = m$ *is a constant, and* $\sigma^2(t) = ct$ *where*

$$c = \frac{1}{T}\int_0^T v^T(t)v(t)\mathrm{d}t. \tag{16}$$

The variance of this Gaussian random variable increases linearly with time, exactly as in a Wiener process.

Remark 6.1 $\alpha(t)$ *becomes, asymptotically with t, a Gaussian random variable with mean* $\mu(t) = m$ *and variance* $\sigma^2(t) = ct$.

Lemma 6.2

$$\mathrm{E}\left[\alpha(t)\alpha(t+\tau)\right] = \begin{cases} \mathrm{E}\left[\alpha^2(t)\right] & \text{if } \tau \ge 0 \\ \mathrm{E}\left[\alpha^2(t+\tau)\right] & \text{if } \tau < 0\end{cases}$$

Corollary 6.1 *Asymptotically with t*

$$\mathrm{E}\left[\alpha(t)\alpha(t+\tau)\right] = m^2 + c\min(t,t+\tau)$$

Definition 6.1 *Two real valued random variables* Ψ_1 *and* Ψ_2 *are called* jointly *Gaussian if for all* $a_1, a_2 \in \mathbb{R}$, *the real random variable* $a_1\Psi_1 + a_2\Psi_2$ *is Gaussian.*

Theorem 6.2 *Asymptotically with time,* $\alpha(t_1)$ *and* $\alpha(t_2)$ *become* jointly *Gaussian.*

The stochastic characterisation of the phase deviation α we obtained in this section can be summarized by Remark 6.1, Lemma 6.2, Corollary 6.1 and Theorem 6.2. These provide adequate information for a practical characterisation of the effect of phase deviation α on the signal generated by an autonomous oscillator, e.g., its spectral properties, as we will see in Section 7 and Section 8.

7 Spectrum of an oscillator with phase noise

Having obtained the asymptotic stochastic characterisation of α, we now compute the power spectral density (PSD) of $x_s(t+\alpha(t))$. We first obtain an expression for the non-stationary autocorrelation function $R(t,\tau)$ of $x_s(t+\alpha(t))$. Next, we demonstrate that the autocorrelation becomes independent of t asymptotically. This implies our main result, that the autocorrelation of the oscillator output with phase noise contains no non-trivial cyclostationary components, confirming the intuitive expectation that a noisy autonomous system cannot have periodic cyclostationary variations because it has no perfect time reference. Finally, we show that the PSD of the stationary component is a summation of Lorentzian spectra, and that a single scalar constant, namely c in (16), is sufficient to characterize it.

We start by calculating the autocorrelation function of $x_s(t+\alpha(t))$, given by

$$R(t,\tau) = \mathrm{E}\left[x_s(t+\alpha(t))x_s^*(t+\tau+\alpha(t+\tau))\right] \tag{17}$$

Definition 7.1 *Define* X_i *to be the Fourier coefficients of* $x_s(t)$: $x_s(t) = \sum_{i=-\infty}^{\infty} X_i \exp(ji\omega_0 t)$.

Lemma 7.1

$$\begin{aligned}R(t,\tau) = \sum_{i=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} X_i X_k^* \exp\left(j(i-k)\omega_0 t\right)\exp\left(-jk\omega_0\tau\right) \\ \mathrm{E}\left[\exp\left(j\omega_0\beta_{ik}(t,\tau)\right)\right]\end{aligned} \tag{18}$$

where $\beta_{ik}(t,\tau) = i\alpha(t) - k\alpha(t+\tau)$.

To evaluate the expectation in the above Lemma, it is useful to consider first the statistics of $\beta_{ik}(t,\tau)$.

Lemma 7.2

$$\lim_{t\to\infty}\mathrm{E}\left[\beta_{ik}(t,\tau)\right] = (i-k)m \tag{19}$$

$$\lim_{t\to\infty}\mathrm{E}\left[\left(\beta_{ik}(t,\tau)\right)^2\right] - \left(\mathrm{E}\left[\beta_{ik}(t,\tau)\right]\right)^2 = \begin{array}{c}(i-k)^2ct + k^2c\tau \\ -2ikc\min(0,\tau)\end{array} \tag{20}$$

where m and c are defined in Theorem 6.1. Also, $\beta_{ik}(t,\tau)$ *becomes Gaussian asymptotically with t.*

Using the asymptotically Gaussian nature of $\beta_{ik}(t,\tau)$, we are now able to obtain a form for the expectation in (18).

Lemma 7.3 *If* $c > 0$, *the characteristic function of* $\beta_{ik}(t,\tau)$ *is asymptotically independent of t and has the following form:*

$$\lim_{t\to\infty}\mathrm{E}\left[\exp\left(j\omega_0\beta_{ik}(t,\tau)\right)\right] = \begin{cases}0 & \text{if } i \ne k \\ \exp\left(-\frac{1}{2}\omega_0^2k^2c|\tau|\right) & \text{if } i = k\end{cases} \tag{21}$$

Lemma 7.4

$$\lim_{t\to\infty}R(t,\tau) = \sum_{i=-\infty}^{\infty} X_i X_i^* \exp\left(-ji\omega_0\tau\right)\exp\left(-\frac{1}{2}\omega_0^2 i^2 c|\tau|\right) \tag{22}$$

The spectrum of $x_s(t+\alpha(t))$ can now be determined as follows:

Lemma 7.5 *The spectrum of* $x_s(t+\alpha(t))$ *is determined by the asymptotic behaviour of* $R(t,\tau)$ *as* $t\to\infty$. *All non-trivial cyclostationary components are zero, while the stationary component of the spectrum is given by:*

$$S(\omega) = \sum_{i=-\infty}^{\infty} X_i X_i^* \frac{\omega_0^2 i^2 c}{\frac{1}{4}\omega_0^4 i^4 c^2 + (\omega + i\omega_0)^2} \tag{23}$$

There is also a term $X_0X_0^*\,\delta(\omega)$ *due to the DC part of* $x_s(t)$, *which is omitted in (23).*

8 Phase noise/timing jitter characterisation

Single-sided spectral density and total power

The PSD $S(\omega)$ in (23) (defined for $-\infty < \omega < \infty$, hence called a double-sided density) is a real and even function of ω, because the periodic steady-state $x_s(t)$ is real hence its Fourier series coefficients X_i in Definition 7.1 satisfy $X_i = X_{-i}^*$. The *single-sided* spectral density (defined for $0 \le f < \infty$) is given by

$$S_{ss}(f) = 2S(2\pi f) = 2\sum_{i=-\infty}^{\infty} X_i X_i^* \frac{f_0^2 i^2 c}{\pi^2 f_0^4 i^4 c^2 + (f + i f_0)^2} \qquad (24)$$

where we substituted $\omega = 2\pi f$ and $\omega_0 = 2\pi f_0$. The *total power* (i.e. the integral of the PSD over the range of the frequencies it is defined for) in $S_{ss}(f)$ is the same as in $S(2\pi f)$, which is

$$P_{tot} = \text{Total power in } S_{ss}(f) = \int_0^{\infty} S_{ss}(f)\,\mathrm{d}f = \sum_{i=1}^{\infty} 2\,|X_i|^2 \qquad (25)$$

Remark 8.1 *The phase deviation $\alpha(t)$ does* not *change the total power in the periodic signal $x_s(t)$, but it alters the power density in frequency, i.e., the power spectral density. For the perfect periodic signal $x_s(t)$, the power spectral density has δ functions located at discrete frequencies (i.e., the harmonics). The phase deviation $\alpha(t)$ spreads the power in these δ functions in the form given in (24), which can be experimentally observed with a spectrum analyzer.*

Single-sideband phase noise spectrum in dBc/Hz

In practice, we are usually interested in the PSD around the first harmonic, i.e., $S_{ss}(f)$ for f around f_0. The *single-sideband* phase noise $\mathcal{L}(f_m)$ (in dBc/Hz) that is very widely used in practice is defined as

$$\mathcal{L}(f_m) = 10\log_{10}\left(\frac{S_{ss}(f_o + f_m)}{2\,|X_1|^2}\right) \qquad (26)$$

For "small" values of c, and for $0 \le f_m \ll f_0$, (26) can be approximated as

$$\mathcal{L}(f_m) \approx 10\log_{10}\left(\frac{f_0^2 c}{\pi^2 f_0^4 c^2 + f_m^2}\right) \qquad (27)$$

Furthermore, for $\pi f_0^2 c \ll f_m \ll f_0$, $\mathcal{L}(f_m)$ can be approximated by

$$\mathcal{L}(f_m) \approx 10\log_{10}\left(\left(\frac{f_0}{f_m}\right)^2 c\right) \qquad (28)$$

Notice that the approximation of $\mathcal{L}(f_m)$ in (28) blows up as $f_m \to 0$. For $0 \le f_m < \pi f_0^2 c$, (28) is not accurate, in which case the approximation in (27) should be used.

Timing jitter

In some applications, such as clock generation and recovery, one is interested in a characterisation of the phase/time deviation $\alpha(t)$ itself rather than the spectrum of $x_s(t + \alpha(t))$ that was calculated in Section 7. In these applications, an oscillator generates a square-wave like waveform to be used as a clock. The effect of the phase deviation $\alpha(t)$ on such a waveform is to create *jitter* in the *zero-crossing* or *transition* times. In Section 6, we found out that $\alpha(t)$ (for an autonomous oscillator) becomes a Gaussian random variable with a linearly increasing variance $\sigma^2(t) = ct$. Let us take one of the transitions (i.e., edges) of a clock signal as a reference (i.e., trigger) transition and synchronize it with $t = 0$. If the clock signal is perfectly periodic, then one will see transitions exactly at $t_k = kT$, $k = 1, 2, \ldots$ where T is the period. For a clock signal with a phase deviation $\alpha(t)$ that has a linearly increasing variance as above, the timing of the kth transition t_k will have a variance (i.e., mean-square error) $\mathrm{E}\left[(t_k - kT)^2\right] = ckT$. The spectral dispersion caused by $\alpha(t)$ in an oscillation signal can be observed with a spectrum analyzer. Similarly, one can observe the timing jitter caused by $\alpha(t)$ using a sampling oscilloscope. McNeill in [20] experimentally observed the linearly increasing variance for the timing of the transitions of a clock signal generated by an autonomous oscillator, as predicted by our theory.

Noise source contributions

The scalar constant c appears in all of the characterisations we discussed above. It is given by

$$c = \frac{1}{T}\int_0^T v_1^T(\tau) B(x_s(\tau)) B^T(x_s(\tau)) v_1(\tau)\,\mathrm{d}\tau \qquad (29)$$

where $B(.) : \mathbb{R}^n \to \mathbb{R}^{n \times p}$ represents the *modulation* of the intensities of the noise sources with the large-signal state. (29) can be rewritten as

$$c = \sum_{i=1}^{p} \frac{1}{T}\int_0^T [v_1^T(\tau) B_i(\tau)]^2\,\mathrm{d}\tau = \sum_{i=1}^{p} c_i \qquad (30)$$

where p is the number of the noise sources, i.e., the column dimension of $B(x_s(.))$, and $B_i(.)$ is the ith column of $B(x_s(.))$ which maps the ith noise source to the equations of the system. Hence, c_i represents the contribution of the ith noise source to c. Thus, the ratio

$$\frac{c_i}{c = \sum_{i=1}^{p} c_i} \qquad (31)$$

can be used as a *figure of merit* representing the contribution of the ith noise source to phase noise/timing jitter.

Phase noise sensitivity

One can define

$$c_s^{(k)} = \frac{1}{T}\int_0^T [v_1^T(\tau) e_k]^2\,\mathrm{d}\tau \qquad (32)$$

(where $1 \le k \le n$ and e_k is the kth unit vector) as the *phase noise/timing jitter sensitivity* of the kth equation (i.e., node), because e_k represents a unit intensity noise source added to the kth equation (i.e., connected to the kth node) in (1).

9 Numerical methods

From Section 6, Section 7 and Section 8, for various phase noise characterisations of an oscillator, one needs to calculate the steady-state periodic solution $x_s(t)$, and the periodic vector $v_1(t)$ in (29). Without providing details, we will present the outline of a time-domain method for computing the periodic vector $v_1(t)$[11]. The procedure for calculating $v_1(t)$ in the time domain is as follows:

1. Compute the large-signal periodic steady-state solution $x_s(t)$ for $0 \le t \le T$ by numerically integrating (1), possibly using a technique such as the shooting method [21].
2. Compute the state-transition matrix $\Phi(T, 0)$ by numerically integrating $\dot{Y} = A(t)Y$, $Y(0) = I_n$ from 0 to T, where the Jacobian $A(t)$ is defined in (3). Note that $\Phi(T, 0) = Y(T)$.
3. Compute $u_1(0)$ using $u_1(0) = \dot{x}_s(0)$. Note that $u_1(0)$ is an eigenvector of $\Phi(T, 0)$ corresponding to the eigenvalue 1.
4. $v_1(0)$ is an eigenvector of $\Phi^T(T, 0)$ corresponding to the eigenvalue 1. To compute $v_1(0)$, first compute an eigenvector of $\Phi^T(T, 0)$ corresponding to the eigenvalue 1, then scale this eigenvector so that $v_1(0)^T u_1(0) = 1$ is satisfied.
5. Compute the periodic vector $v_1(t)$ for $0 \le t \le T$ by numerically solving the adjoint system

$$\dot{y} = -A^T(t) y \qquad (33)$$

using $v_1(0) = v_1(T)$ as the initial condition. Note that $v_1(t)$ is a periodic steady-state solution of (33) corresponding to the Floquet exponent that is equal to 0, i.e., $\mu_1 = 0$. It is not possible to calculate $v_1(t)$ by numerically integrating (33) *forward* in time, because the numerical errors in computing the solution and the numerical errors in the initial condition $v_1(0)$ will excite the *modes* of the solution of (33) that grow without bound. However, one can integrate (33) *backwards* in time with the "initial" condition $v_1(T) = v_1(0)$ to calculate $v_1(t)$ for $0 \le t \le T$ in a numerically stable way.

6. Then, c is calculated using (29).

[11] We also developed a frequency domain numerical method based on an harmonic balance formulation.

0 Examples

)scillator with a bandpass filter and a nonlinearity [22]

'his oscillator (Figure 1) consists of a Tow-Thomas second-order andpass filter and a comparator [22]. If the OpAmps are considered to e ideal, it can be shown that this oscillator is equivalent (in the sense f the differential equations that describe it) to a parallel RLC circuit in arallel with a nonlinear voltage-controlled current source (or equiva- :ntly a series RLC circuit in series with a nonlinear current-controlled oltage source). In [22], authors breadboarded this circuit with an ex- :rnal white noise source (intensity of which was chosen such that its ffect is much larger than the other internal noise sources), and mea- ɹred the PSD of the output with a spectrum analyzer. For $Q = 1$ and $_o = 6.66$ kHz, we performed a phase noise characterisation of this scillator using our numerical methods, and computed the periodic os- illation waveform $x_s(t)$ for the output and $c = 7.56 \times 10^{-8}$ sec^2.Hz. 'igure 2(a) shows the PSD of the oscillator output computed using 24), and Figure 2(b) shows the spectrum analyzer measurement[12]. 'he single-sideband phase noise spectrum using both (27) and (28) ; in Figure 3 . Note that (28) can not predict the PSD accurately be- ɔw the cut-off frequency $f_c = \pi f_0^2 c = 10.56$ Hz (marked with a $*$ in 'igure 3) of the Lorentzian.

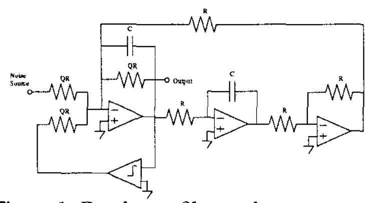
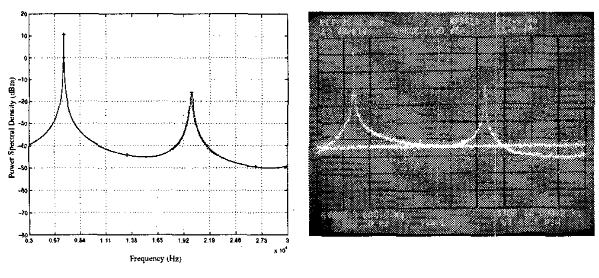


Figure 1: Band-pass filter and a comparator



(a) Computed PSD (4 harmonics) (b) Measured PSD [22]

Figure 2: Computed and measured PSD

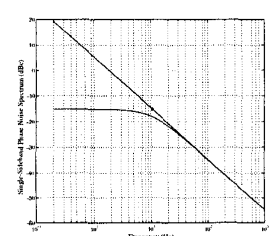


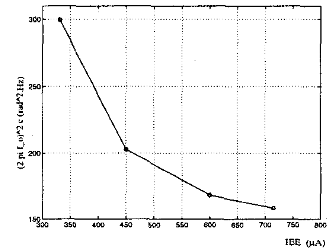
Figure 3: $\mathcal{L}(f_m)$ computed with both (27) and (28)

Ring oscillator

The ring oscillator circuit is a three stage oscillator with fully differential ECL buffer delay cells (differential pairs followed by emitter followers). This circuit is from [20]. [20] and [23] use analytical techniques to characterize the timing jitter/phase noise performance of ring-oscillators with ECL type delay cells. Since they use analytical techniques, they use a simplified model of the circuit and make several approximations in their analysis. [20] and [23] use time-domain Monte Carlo noise simulations to verify the results of their analytical results. They obtain qualitative and some quantitative results, and offer guidelines for the design of low phase noise ring-oscillators with ECL type delay cells. However, their results are only valid for their specific oscillator circuits. We will compare their results with the results we will obtain for the above ring-oscillator using the general phase noise characterisation methodology we have proposed which makes it possible to analyze a complicated oscillator circuit without simplifications. We performed several phase noise characterisations of the bipolar ring-oscillator. The results are shown in Figure 4(a), where R_c is the collector load resistance for the differential pair (DP) in the delay cell, r_b is the zero bias base resistance for the BJTs in the DP, I_{EE} is the tail bias current for the DP, and f_o is the oscillation frequency for the three stage ring-oscillator. Note that the changes in R_c and r_b affect the oscillation frequency, unlike the changes in I_{EE}. Figure 4(b) shows a plot of $(2\pi f_o)^2\, c$ versus I_{EE} using the data from Figure 4(a). This prediction of the dependence of phase noise/timing jitter performance on the tail bias current is in agreement with the analysis and experimental results presented in [20] and [23] for ring-oscillators with ECL type delay cells. Note that larger values for $(2\pi f_o)^2\, c$ indicate *worse* phase noise performance.

R_c (Ω)	r_b (Ω)	I_{EE} (μA)	f_o (MHz)	c (sec^2.Hz $\times 10^{-15}$)
500	58	331	167.7	0.269
2000	58	331	74	0.149
500	1650	331	94.6	0.686
500	58	450	169.5	0.182
500	58	600	169.7	0.151
500	58	715	167.7	0.142

(a) Phase noise characterisation



(b) Phase noise performance versus I_{EE}

Figure 4: Ring-oscillator

[12]The PSDs are plotted in units of dBm.

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