Fast and Accurate Simulation of Coupled Oscillators Using Nonlinear Phase Macromodels

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***Abstract*— We present a generally applicable, macromodel-based method for predicting the behaviour of coupled oscillators much faster than possible using SPICE-level simulation. Our method is based on the use of nonlinear phase and amplitude macromodels that are extracted automatically from oscillators' SPICE-level circuits. Using these macromodels in place of the original SPICE-level circuits enables large speedups, while capturing phase and amplitude dynamics, during simulation of synchronous and asynchronous phenomena in coupled oscillators. We demonstrate the technique on coupled Colpitts oscillators, obtaining speedups of 70× without loss of accuracy. Greater speedups are expected when the size of coupled systems is larger and oscillators are more complex. Our technique is easily extended to simulate coupled oscillator networks comprising of large number of oscillators with different operating mechanisms.**

I. Introduction

Coupled nonlinear oscillators model a wide variety of physical systems. For example, in microwave systems, a quasi-optical power-combining technique [1], [2], which exploits coupled oscillators, has been proposed for realizing high-power RF sources using solid-state devices. In biological and medical applications, coupled oscillator models are used for mathematical modeling of, for example, electrical activity in cardiac tissues [3], the synchronized flashing of systems of fireflies , neuronal systems and so on. Indeed, coupled oscillator models arise in contexts ranging from the nanoscale (*e.g.*, Josephson junction arrays, spintronics) to the cosmic scale (*e.g.*, gravitational interactions between systems of galaxies). It is not surprising that fast and accurate simulation of the dynamics of coupled oscillators is of significant theoretical and practical interest in a multitude of disciplines.

In the context of circuits, the traditional method for simulating nonlinear systems is to use simulators such as SPICE . However, SPICE-level simulation is far from ideally suited for simulating oscillators in general, and especially for coupled oscillators. The pull-in process of coupled oscillators involves strongly nonlinear phase and amplitude dynamics, spread over widely separated time scales. Because transient simulation algorithms fundamentally accumulate numerical phase errors unboundedly for oscillators, this results in severe accuracy problems. To improve simulation accuracy, very many timesteps have to be taken in each oscillation cycle, resulting in additional simulation inefficiency that exacerbates the inefficiency already associated with simulating systems with widely separated time scales of activity.

To overcome the limitations of transient simulation, various analytical approaches [4]–[7] have been proposed. In [5], the authors showed that two resistively-coupled oscillators synchronize with a 180° phase shift. In [6], the authors presented an analytical investigation of the synchronous dynamics of two coupled multivibrators; while in [7], an analytical approach was developed that is capable of considering both synchronous and asynchronous dynamics in coupled relaxation oscillators. Such analytical techniques are valuable in that they provide essential intuition and understanding of how and why specific coupled oscillator systems behave as they do.

From a computational standpoint, however, such detailed studies are of limited value, since they are overly specialized for specific oscillator systems, which are, moreover, often simplified for analytical tractability. However, in real applications, coupled oscillator systems can be very large and complex, and qualitatively important collective phenomena can be lost if simplified models are used. For example, in the simulation of the human heart, it is necessary to model networks with at least tens of thousands of units, with each unit being a complex oscillator with tens of nonlinear equations [3]. Hence, a *general computational method* that can apply to large-scale systems of coupled oscillators is of great interest.

In this paper, we present a numerical approach to predict synchronous and asynchronous phenomena in coupled nonlinear oscillators. Our method is based on replacing the actual oscillators being coupled (which can be large) by *small, accurate, nonlinear macromodels* that behave very similarly. Instead of simulating the full circuit, we simulate a simple phase equation, resulting in dramatic improvements in simulation efficiency.

Although a number of linear oscillator phase models (*e.g.*, [8]–[10]) have been proposed for predicting phase information, it has been shown [11] that linear models cannot capture nonlinear phase dynamics. In particular, linear phase macromodels can be proven not to capture *injection locking*, a phenomenon crucial in the interaction of coupled oscillators. In contrast, the rigorous nonlinear phase macromodel [12] we adopt in this work is indeed able [11] to capture injection locking in oscillators.

The nonlinear phase macromodels we use depend centrally on a quantity called the *Perturbation Projection Vector* (PPV). This is a periodic vector function of time, specific to any given oscillator, that can be uniquely determined via efficient numerical methods [12]–[15]. Once the PPV-based nonlinear macromodels have been obtained, we substitute them for the oscillators and represent the couplings between separate oscillators as perturbations. Via this procedure, we obtain a much smaller system which is, moreover, in the phase domain, hence much more efficient to simulate than the original oscillator system. In addition, by incorporating a few amplitude modes as well into the phase-domain macromodel [16], we are further able to predict a much wider range of coupled-oscillator phenomena than with the phase macromodels alone. In particular, the effects of loading of each oscillator by the coupling network, which can be important in some situations (such as when coupled oscillators are not synchronous) are predicted very well when amplitude components are incorporated into the macromodel. We emphasize again that this technique is generally applicable to *any* kind of oscillator (so long as it can be described by a system of differential equations) and scales transparently to large coupled oscillator networks.

We demonstrate our technique on coupled Colpitts LC oscillators and compare simulation results obtained against the original oscillator circuits (*i.e.*, full SPICE-level circuit simulation). We show that our macromodel-based technique is able to capture both synchronous and asynchronous phenomena in coupled oscillators. Even with the relatively small oscillator circuits we have used for testing purposes, we obtain speedups in the range of 1-2 orders of magnitude; we expect much greater speedups for larger coupled networks with complex unit equations.

The remainder of the paper is organized as follows. In Section II, we review the nonlinear oscillator macromodel we employ in this work. In Section III, we describe our nonlinear phase macromodel-based technique for simulating coupled oscillators. In Section IV, we present simulation results on coupled Colpitts oscillators.

II. Nonlinear Oscillator Macromodel

A general oscillator under perturbations can be described by

$$\dot{x} + f(x) = Bb(t), \tag{1}$$

where $b(t)$ is perturbation signal applied to the free running oscillator.

According to [12], the solution of the oscillator under perturbation can be expressed as

$$x(t) = x_s(t + \alpha(t)) + y(t), \tag{2}$$

where $x_s(t)$ is the steady-state orbit of the unperturbed oscillator, $x(t)$ is the orbit of the perturbed oscillator, $\alpha(t)$ is the phase deviation of the oscillator due to perturbation, and $y(t)$ is the oscillator's amplitude variations.

So the orbital deviation of the oscillator can be decomposed into two parts: phase deviation $\alpha(t)$ and amplitude variations $y(t)$. If we can calculate the phase deviation $\alpha(t)$ and amplitude variations $y(t)$, we can rebuild the waveforms of the oscillator under perturbations using (2).

A. Nonlinear Oscillator Phase Macromodel

As proven in [12], The phase deviation $\alpha(t)$ is governed by a simple one-dimensional nonlinear differential equation

$$\dot{\alpha}(t) = v_1^T(t + \alpha(t)) \cdot Bb(t), \tag{3}$$

where $b(t)$ is perturbation signals applied to the oscillator and $v_1(t)$ is the perturbation projection vector (PPV) representing the oscillator's phase sensitivity to perturbations. The PPV has the periodic waveforms which have the same frequency as that of the oscillator. Various methods [12]–[15], both in time domain and frequency domain, have been presented for calculating the PPV from SPICE-level circuit descriptions of oscillators.

In (3), $v_1(t)$ is a vector: each entry in $v_1(t)$ represents the phase sensitivity of the corresponding circuit node. Hence, we can simulate the phase of the oscillator under multiple perturbations by solving this simple scalar equation.

B. Amplitude Macromodel

Since the phase deviation of oscillators can be obtained by solving (3), we can calculate amplitude variations of oscillators by linearizing the oscillators over their perturbed time-shifted orbits $x_s(t + \alpha(t))$. In [16], a method is presented to to construct amplitude macromodels of oscillators. the oscillator is first linearized on $x_s(t + \alpha(t))$

$$\begin{aligned} \dot{o}(t) &\approx -\frac{\partial f}{\partial x}\Big|_{x_s(t+\alpha(t))} o(t) + b(t) \\ &= A(x_s(t + \alpha(t)))o(t) + b(t), \end{aligned} \tag{4}$$

where $x_s(t)$ is the oscillator's steady-state orbit, $o(t)$ is the small variations due to perturbation $b(t)$. By introducing a new variable $\hat{t} = t + \alpha(t)$ and defining $\hat{o}(\hat{t}) = o(t)$ and $\hat{b}(\hat{t}) = b(t)$, we can get a linear periodic time-varying (LPTV) system

$$\dot{\hat{o}}(\hat{t}) = A(x_s(\hat{t}))\hat{o}(\hat{t}) + \hat{b}(\hat{t}). \tag{5}$$

Applying Floquet decomposition, the LPTV system can be decomposed into a diagonalized LTI system with periodic input/output vectors

$$\hat{o}(\hat{t}) = \sum_{i=1}^{n} u_i(\hat{t}) \int_0^{\hat{t}} \exp(\mu_i(\hat{t} - \tau)) v_i^T(\tau)\hat{b}(\tau)d\tau, \tag{6}$$

where μ_i are Floquet exponents, and $v_i(t)$ and $u_i(t)$ are periodic input/output vectors. By dropping the Floquet exponent corresponding to phase and other less important Floquet exponents, we can get a reduced amplitude macromodel.

III. Simulating Coupled Oscillators Using Nonlinear Oscillator Macromodel

Once the macromodel of the oscillator is obtained, the method for simulating coupled oscillators using nonlinear oscillator macromodel is quite straightforward. Here, we give a simple example for this method on a system of 2 oscillators with resistor coupling.

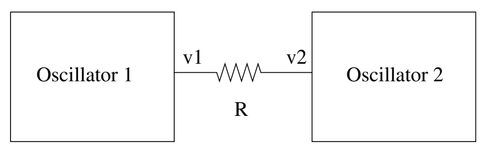

Fig. 1. Block diagram of coupled oscillators.

The block diagram of the circuit is shown in Figure 1, where R is the resistance of the coupling resistor, v_1 and v_2 are corresponding node voltages on both sides of the resistor.

First, we need to extract the PPV, or the phase sensitivity, of the oscillators. When the PPV is available, the phase of the coupled oscillators can be simulated by solving following equations

$$\boxed{\begin{cases} b(t) = \dfrac{v_1(t) - v_2(t)}{R} \\ \dot{\alpha}_1(t) = v_{1_{osc1}}^T(t + \alpha_1(t)) \cdot Bb(t) \\ \dot{\alpha}_2(t) = v_{1_{osc2}}^T(t + \alpha_2(t)) \cdot B(-b(t)), \end{cases}} \tag{7}$$

where $\alpha_1(t)$ and $\alpha_2(t)$ are phase deviations of coupled oscillators, $v_{1_{osc1}}(t)$ and $v_{1_{osc2}}(t)$ are PPVs, and $v_1(t)$ and $v_2(t)$ are voltages swinging on both sides of the coupling resistor, which can be calculated using (2). If the amplitude variations are very small, which likely happens when the coupling is weak, we can ignore $y(t)$ in (2) and calculate the voltages using (8)

$$x(t) = x_s(t + \alpha(t)). \tag{8}$$

If amplitude variations are large and cannot be ignore, we can use (6) to calculate amplitude variations $y(t)$ and reproduce the waveforms of v_1 and v_2 using (2).

Since the nonlinear phase equation (3) accepts multiple inputs, we can easily extend this method to larger networks with more complex couplings. For each oscillator in the network, we just need to add one phase equation, which handles all couplings to this oscillator, to our macromodelled system. The resulting macromodelled system have much smaller system size, and can be simulated more efficiently than the original circuit.

IV. Numerical Results

In this section, we evaluate the technique presented above using coupled Colpitts LC oscillators. All simulations were performed using **MATLAB** on an Linux system. We constructed oscillator macromodels using the method described in Section II, simulated the phase of the coupled oscillators, and compared the results with SPICE-like simulations of the full oscillator circuits. Experiment results show that our method is able to capture both synchronous and asynchronous phenomena in coupled oscillators accurately, with 70 times speedups.

Our test circuit is a coupled system of 2 Colpitts LC oscillators coupled by voltage controlled current source. The circuit and parameters of the Colpitts LC oscillator are shown in Figure 2. The oscillator has a system size of 6 and a free-running frequency of $f_0 = 4GHz$. The coupled system is shown in Figure 3, where 'Oscillator2' has a free-running frequency of $f_0 = 4GHz$ and 'Oscillator1' has a free-running frequency of $0.988f_0$, 1.2% slower than 'Oscillator2'. The coupling strength can be adjusted by changing the conductance g_m.

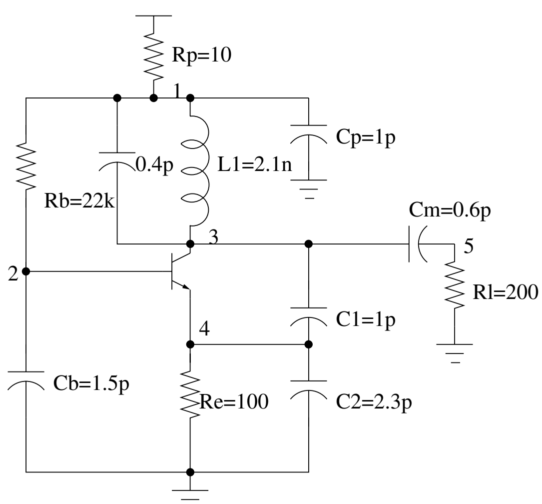

Fig. 2. A 4GHz Colpitts LC oscillator.

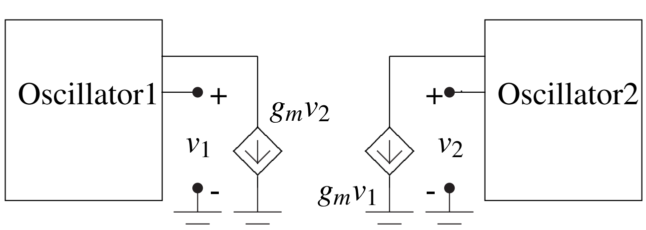

Fig. 3. Block diagram of the coupled system.

First, we choose coupling conductance $g_m = \frac{1}{4000}$, and run SPICE-level simulation on the full circuit. After hundreds of oscillation cycles, the coupled system synchronize to a common frequency, as shown in Figure 4. We can see from Figure 4 that these two oscillators synchronize very well, with a constant phase difference. The measured common frequency of these two oscillators is about $0.995f_0$, which means 'Oscillator1' runs 0.7% faster and 'Oscillator2' runs 0.5% slower after the coupling.

We then simulate the phase of the coupled system directly using our macromodelled system, and plot the simulation results in Figure 5. From Figure 5, we can see that the time shift of 'Oscillator1' grows with a constant slope of 0.68%, which implies that 'Oscillator1' runs 0.68% faster after the coupling. Similarly, the time shift of 'Oscillator1' has a constant slope of -0.53% in Figure 5, which means the frequency of 'Oscillator2' decreases with a percentage of 0.53%. These results match full simulation very well, hence, our nonlinear macromodel can predict the synchronous phenomenon in coupled oscillators accurately.

In this simulation, the SPICE-level full simulation takes about 24 minutes for a simulation time of 400 cycles; however, it takes only about 20 seconds to simulate the same number of cycles using our macromodel. This gives us approximately 70 times speedup on this small coupled system.

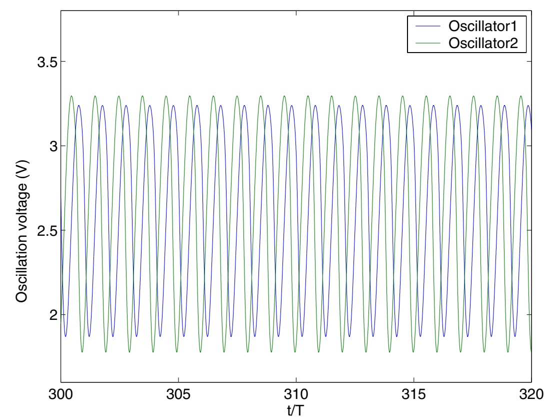

Fig. 4. Full simulation waveforms of the coupled oscillators ($g_m = \frac{1}{4000}$).

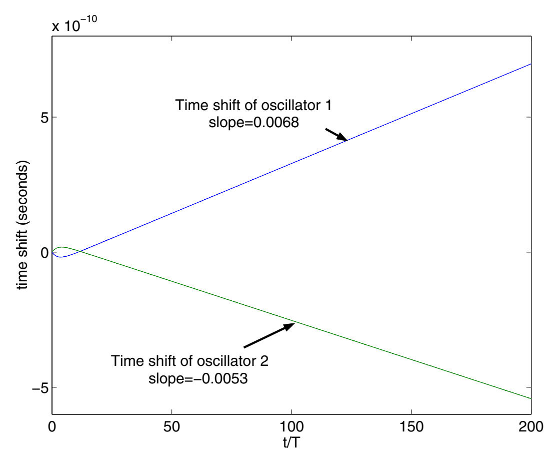

Fig. 5. Time shift of the coupled oscillators from macromodel($g_m = \frac{1}{4000}$).

Now we decrease the coupling conductance to $g_m = \frac{1}{20000}$, which implies a very weak coupling. We run the full simulation, and plot the output waveform of 'Oscillator1' in Figure 6. It is obvious that these two oscillators haven't locked to a common frequency, resulting periodic beat notes in the output waveforms.

We then simulate the phase of the coupled system directly using our macromodel. We plot the simulation results in Figure 7 and Figure 8. For comparison purpose, we use a program to extract time shifts from full simulation waveforms, and plot them together with the results from our macromodel. Both results indicate that the time shift of the oscillator changes without a constant slope when the coupled system is not in lock. Our macromodel can predict the time shift of the asynchronous coupled oscillator accurately. There has slight difference between full simulation and macromodel in Figure 7, which can be due to the extraction error, because it is not accurate to get phase information from time domain waveforms.

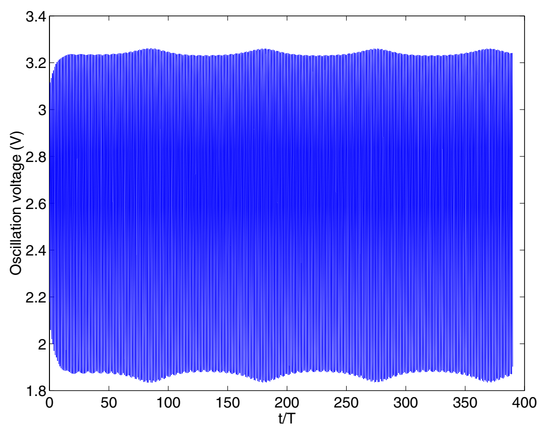

Fig. 6. Full simulation waveforms of the coupled oscillators ($g_m = \frac{1}{20000}$).

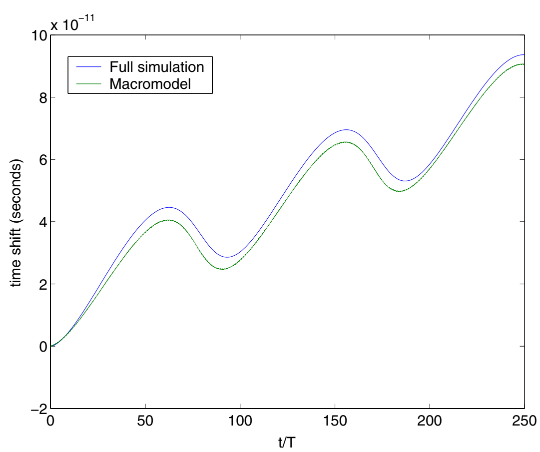

Fig. 7. Time shift of 'Oscillator1'($g_m = \frac{1}{20000}$).

V. Conclusions

We have presented a nonlinear macromodel-based technique for predicting behaviors of coupled oscillators. Our method is a general approach, applicable to any oscillators. Moreover, our method has great scalability, can be extended to analyze large scale oscillator networks. The technique we presented can capture the synchronous and asynchronous phenomena in coupled oscillators accurately, with great speedups over full SPICE-level simulation. Simulation results show 70× speedup obtained, without loss of simulation accuracy. Currently, we are working on incorporating amplitude macromodels to rebuild waveforms of asynchronous couple oscillators.

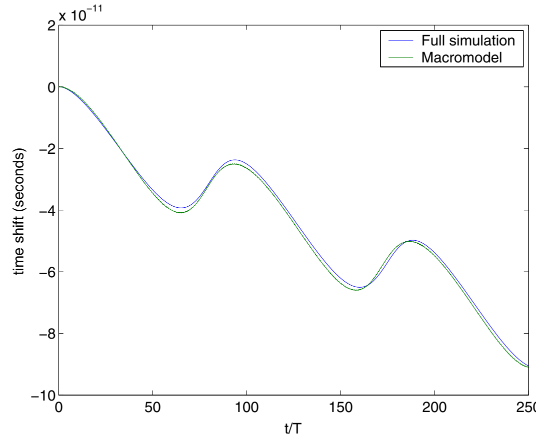

Fig. 8. Time shift of 'Oscillator2'($g_m = \frac{1}{20000}$).

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