Modeling and Analysis of (Nonstationary) Low Frequency Noise in Nano Devices: A Synergistic Approach based on Stochastic Chemical Kinetics

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***Abstract*—Defects or traps in semiconductors and nano devices that randomly capture and emit charge carriers result in low-frequency noise, such as burst and $1/f$ noise, that are great concerns in the design of both analog and digital circuits. The capture and emission rates of these traps are functions of the time-varying voltages across the device, resulting in nonstationary noise characteristics. Modeling of low-frequency, nonstationary noise in circuit simulators is a long-standing open problem. It has been realized that the low frequency noise models in circuit simulators were the culprits that produced erroneous noise performance results for circuits under strongly time-varying bias conditions. In this paper, we first identify an almost perfect analogy between trap noise in nano devices and the so-called ion channel noise in biological nerve cells, and propose a new approach to modeling and analysis of low-frequency noise that is founded on this connection. We derive two fully nonstationary models for traps, a fine-grained Markov chain model based on recent previous work and a completely novel coarse-grained Langevin model based on similar models for ion channels in neurons. The nonstationary trap models we derive subsume and unify all of the work that has been done recently in the device modeling and circuit design literature on modeling nonstationary trap noise. We also describe joint noise analysis paradigms for a nonlinear circuit and a number of traps. We have implemented the proposed techniques in a Matlab® based circuit simulator, by expanding the industry standard compact MOSFET model PSP to include a nonstationary description of oxide traps. We present results obtained by this extended model and the proposed simulation techniques for the low frequency noise characterization of a common source amplifier and the phase jitter of a ring oscillator.**

***Index Terms*—low frequency noise, RTS noise, nonstationary noise, noise analysis, Langevin equation, stochastic chemical kinetics**

I. INTRODUCTION

A. RTS Noise in Transistors and Electronic Circuits

In semiconductors, defects or traps in the crystal structure can randomly capture and emit charge carriers, resulting in fluctuations in the number of mobile charge carriers, which can subsequently cause noise in electric fields, currents and voltages. In MOSFETs, the traps in the gate oxide are widely believed to be the source of various kinds of low-frequency noise, such as popcorn or burst noise and $1/f$ noise [1]. The trap occupancy function, indicating whether the trap is full or not, is usually modeled as a two-level Random Telegraph Signal (RTS) [2]. Hence, the noise due to the random trapping of charge carriers in semiconductors and transistors is usually referred to as RTS noise [3]. Low-frequency and RTS noise is a major concern in the design of electronic circuits used in RF applications (low-noise amplifiers, mixers, oscillators, PLLs, etc.) and CMOS image sensors [4], [5], [6]. Moreover, due to technology scaling, low-frequency noise in MOSFETs is emerging as a source of great concern in the design of even digital circuits such as SRAMs and DRAMs [7], [8].

B. Modeling and Analysis of Low Frequency Noise in Circuits

Modeling of low-frequency and $1/f$ noise in MOSFETs, a subject with rampant speculations and a great deal of controversy, has been a preoccupation for researchers for many decades [9], [10], [11], [12], [13], [14], [15], [16]. However, almost all of this effort was directed at modeling low-frequency noise in transistors at a constant bias point, with *stationary* statistics for the noise source. Stationary, frequency domain models for $1/f$ noise was incorporated into circuit simulators such as SPICE in the very early days [17], [18] and frequency domain, AC noise analysis has been a workhorse tool in analog circuit design since then.

With the explosive growth of wireless mobile communications in the 90's, circuits for RF applications such as mixers, oscillators and PLLs have become an essential part of electronic systems and consumer electronic devices. Analysis of RF circuits posed a great challenge to circuit simulators due to their highly nonlinear nature and the wide frequency range of operation. Since these circuits did not operate in the small-signal regime, noise models for transistors based on quiescent conditions and frequency domain AC noise analysis based on linear time-invariant (LTI) system theory and stationary stochastic processes were deemed not applicable [15]. As a result, a great deal of work was done in the 90's in developing nonstationary noise models for transistors and noise analysis techniques that can handle time-varying bias conditions [15], [19], [20], [21], [22], [23], [24]. This work resulted in specialized circuit noise analysis techniques that were then implemented in RF circuit simulators. However, the modeling of (nonstationary) low-frequency and $1/f$ noise under time-varying bias conditions has remained as an open problem [15], [25], [26], [27]. Just when analog/RF electronic circuit designers started using the newly developed RF circuit noise analysis techniques, researchers reported that these analyses implemented in circuit simulators mispredicted the impact of low-frequency noise on the performance of oscillators and CMOS image sensor circuitry under strongly time-varying bias conditions and blamed the low-frequency noise models as the culprit, see e.g. [28], [4], [5], [3]. Subsequently, starting with the seminal paper by El Gamal *et.al.* [5], substantial amount of work has been done in developing correct, truly nonstationary low-frequency noise models, see e.g. [4], [5], [3], [29], [30]. However, it seems that the results of this work have not been incorporated, to date, into previously developed RF circuit noise analysis techniques and RF circuit simulators.

C. Membrane Ion Channel Noise in Nerve Cells

Nerve cells (also known as neurons), special types of cells that make up the brain and the nervous system, process and transmit information based on electrical and chemical signaling. Neurons are able to maintain a voltage difference between the cell interior and exterior by pumping charged ions (such as sodium and potassium) through *ion channels* in their membranes and hence creating an ion concentration difference. Ion channels are made up of proteins that can open and close and hence allow or block the passage of ions. The state of some ion channels, so-called *voltage-gated* channels, is a function of the (trans) membrane voltage. Other so-called *ligand-gated* channels are controlled by the binding of a substance. The operation and dynamical behavior of neuron ion channels and currents

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This work was supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK) under project 111E188.

978-1-4799-1071-7/13/$31.00 ©2013 IEEE

and the membrane voltage was first laid out by Hodgkin and Huxley in 1952 [31]. It was later realized that individual ion channels behave stochastically, by closing and opening at random times [32], but not in a totally arbitrary manner. For voltage-gated channels, the rates at which they close and open is a function of the membrane voltage. The stochastic ion channel behavior has been identified as a source of electrical *channel noise* and fluctuations in the membrane voltage [32]. Channel noise in neurons is believed to both influence the reliability of nerve cell responses in a detrimental manner but at the same time enable a "broader repertoire of neuronal behavior" [32]. Ion channel noise is claimed to be one of the determining factors in the size of a brain [33].

D. A Perfect Analogy

Given the description of the charge carrier traps in semiconductors and transistors and the ion channels in nerve cell membranes, the following analogy can be formed between them. Both traps and ion channels behave stochastically and are in one of two states, empty/full for traps and open/closed for ion channels. The rates at which traps capture and emit charge carriers are in fact a function of the voltages across the MOSFET [5], [3], very similar to the case of voltage-gated ion channels. Stochastic behavior of ion channels results in a noisy ion channel current which subsequently causes noise in the membrane voltage. Similarly, the "channel" current in a MOSFET becomes noisy due to the random emission and capture of charge carriers by the traps which then results in noisy voltages. Furthermore, both traps in transistors and ion channels in neurons behave independently of each other in a stochastic sense, but the transition rates of all of the ion channels in a particular nerve cell membrane and the traps in a particular transistor are a function of the same voltage, the membrane voltage in one case and the voltage applied across the transistor in the other. Even though the traps and ion channels are stochastically independent of each other, the same voltage that controls the transition rates creates a sort of global coupling [34].

As electronic circuit models are commonly used for neurons and neuronal networks, the almost perfect and complete analogy described above becomes very natural and can be extended to encompass the noisy behavior of networks of neurons on one side and networks of transistors on the other. A neuron is the most basic building block of a biological neuronal network or the nervous system, just like the transistor for a microelectronic system. It turns out that exactly the same mathematical model, i.e., an asymmetrical two-level RTS signal corresponding to a simple two-state Markov chain, has been used in modeling the stochastic behavior of both traps [5], [3] and ion channels [32]. The voltage dependence of transition rates in this RTS model in both cases is the source of intricate nonstationary behavior, which makes the modeling and analysis of the resulting noise and its impact on the system performance a great challenge.

E. Modeling and Analysis of Noise in the Nervous System

The deterministic dynamical behavior of neurons has long been analyzed based on the celebrated Hodgkin-Huxley equations, a small set of nonlinear differential equations describing the dynamics of the membrane voltage and the ion channels, commonly interpreted as an electronic circuit composed of a nonlinear capacitor and nonlinear, voltage-controlled conductances [31], [34], [35]. Modeling of ion channel noise in neurons is typically done by a stochastic modification/extension of the Hodgkin-Huxley equations [36], [37], [38], [39], [40], [34], [41], [32]. The so-called exact method is based on modeling every ion channel as a two-state, continuous-time, discrete-space Markov chain, resulting in a stochastic automaton model [34]. Then, the differential equations for the membrane voltage from the Hodgkin-Huxley model and the Markov chains for the individual ion channels are simulated together in a Monte Carlo manner using a hybrid variant of the Stochastic Simulation Algorithm (SSA) [42], [43], [44], [45]. SSA was first proposed as a Monte Carlo algorithm for simulating coupled chemical reactions, where a molecule count based discrete-space model is used to represent the amount of chemical reactants. In hybrid variants of SSA [45], a continuous-space, differential equation model is used for some parts of the system while the rest is modeled in discrete-space. While the simulation algorithm keeps track of (i.e., generates a sample path for) the states of all of the ion channels, the differential equations for the membrane voltage are integrated with an appropriate numerical technique where the channel conductances are determined by the states of the ion channels. Since this is a computationally costly algorithm, simpler variants which keep track not of the states of every ion channel but only the total number of channels in each state were also used [32]. This type of hybrid Monte Carlo simulation was deemed to be still too expensive and approximate models where channel noise is represented by continuous-space, stochastic differential equations (SDE), i.e., Langevin equations [46], were then proposed [34], [41], [32]. This Langevin approach essentially amounts to introducing stochastic noise terms into the original Hodgkin-Huxley equations, resulting in a system of nonlinear SDEs, which are then solved numerically with an appropriate technique, again in a Monte Carlo manner to generate sample paths for the noisy membrane voltage [34], [41]. Currently, there seems to be an unsettled controversy in the channel noise modeling literature [37], [38], [39], [40], [34], [41], [32] on the accuracy of the continuous Langevin channel noise model against the discrete Markov model.

F. Summary of Contributions, Outline of the Paper, Applications

In this paper, we adopt a synergistic approach in developing noise modeling and analysis techniques for RTS and low-frequency noise in transistors and electronic circuits based on techniques for modeling ion channel noise in neurons.

We first develop stochastic, nonstationary models for charge carrier traps in semiconductors, i.e., RTS noise, by benefiting from the two decades of ion channel noise modeling and simulation work in neurobiology [32] and the extensive literature on modeling and simulation of stochastic chemical kinetics [43], [44]. In particular, we develop (i) a continuous-time, discrete-space, fine-grained Markov chain model in Section II-B (previously known [8]), and (ii) a continuous-time, continuous-space, coarse-grained Langevin stochastic model (completely new) in Section II-C for a charge carrier trap, that are both fully nonstationary capturing the impact of arbitrary time-varying bias conditions. The Langevin model we develop, even though approximate and coarse-grained, correctly and *exactly* captures the first (mean behavior) and second-order (autocorrelation and spectral properties) probabilistic characteristics of a trap. In most, but possibly not all, applications in electronic design, such a probabilistic characterization is adequate.

We discuss, in Section III, how to incorporate the above stochastic trap models in circuit simulation, i.e., describe techniques for the *joint* analysis and simulation of an electronic circuit and a number of traps. In particular, we describe (i) a stochastic, hybrid, Monte Carlo type simulation algorithm, in Section III-B, for the joint simulation of a circuit described by a set of nonlinear differential equations and a number of traps represented by the fine-grained Markov chain model, (ii) a stochastic, Monte Carlo type simulation algorithm, in Section III-C, for the joint simulation of a circuit described by a set of nonlinear differential equations and a number of traps represented by the coarse-grained Langevin model.

The applications of the nonstationary trap models and noise analysis techniques outlined above include noise analysis for RF circuits [15], [20], [21], [22], [23], [24], CMOS image sensors [4], [47], [6], SRAMs and DRAMs [7], [8], and phase noise analysis for oscillators [20], [27]. In Section IV, we present results obtained by the proposed models and the simulation techniques for the low frequency noise characterization of a common source amplifier and the phase jitter of a ring oscillator.

Finally, we note that the configuration and locations of the traps in a transistor are not known ahead of time and determined randomly at fabrication [48]. Naturally, the trap configurations for different transistors are not the same. As such, one has to not only deal with the temporal noise caused by the traps but also with the statistical aspect of the problem due to trap configurations that do not change over time but determined randomly at fabrication. In this paper, we do not discuss this important aspect of the problem in hand and concentrate only on the analysis of temporal noise caused by a given configuration of traps and its impact on circuit performance.

II. NONSTATIONARY STOCHASTIC TRAP MODEL

A. Preliminaries

Let $N(t)$ be the *occupancy function* for a certain trap, i.e., $N(t) = 1$ if the trap is full and $N(t) = 0$ if empty. $N(t)$ is modeled as a two-state, continuous-time Markov chain that is characterized by the capture rate $\lambda_c(t)$ and the emission rate $\lambda_e(t)$. The time dependence of the capture/emission rates stem from the fact that these rates depend on the voltages across the MOSFET which can considerably vary with time during large-signal operation. For notational simplicity, we do not express the capture/emission rates as an explicit function of these voltages but simply denote their time dependence. As a result, the trap occupancy function $N(t)$ must be regarded as an *inhomogeneous* Markov chain. Moreover, due to noise in the circuit, the voltages across the MOSFET must be treated as noisy signals and hence represented as stochastic processes. This means that the capture/emission rates are also stochastic, making $N(t)$ a *doubly stochastic* Markov chain [49]. We will deal with the doubly stochastic nature of $N(t)$ and its implications later in our treatment.

B. Chapman-Kolmogorov Equation for a Trap

Let $\mathbb{P}(n,t)$ denote the probability that $N(t) = n$, where n is either 0 (empty) or 1 (full). Given that we know the probability that the trap is in any one of the possible states at time t, we will derive an equation for the trap to be in a certain state n at time $t + dt$, where dt is assumed to be small enough so that at most one capture or emission event can occur in the time interval $[t, t + dt)$ [44]. For the trap to be in state n at time $t + dt$, either the trap was already in state n at time t and no event has occurred in $[t, t + dt)$, or the trap was in some other state n' at time t and an event has occurred in $[t, t + dt)$ which caused the state to change from n' to n. Let A be the event that the trap is in state n at time $t + dt$. Let $H_\emptyset$, H_c and H_e be the events that the trap is in state n, $n - 1$ and $n + 1$ (at time t) respectively. In this case, the conditional probabilities $\mathbb{P}(A\,|H_c)$ and $\mathbb{P}(A\,|H_e)$ are in fact the probabilities of the capture and emission events occurring in $[t, t + dt)$ [44]. Due to the Markov property and based on the capture and emission rates $\lambda_c(t)$ and $\lambda_e(t)$ [50], these conditional probabilities can be expressed as follows

$$\mathbb{P}(A\,|H_c) \;=\; \lambda_c(t)\,dt \tag{1}$$

$$\mathbb{P}(A\,|H_e) \;=\; \lambda_e(t)\,dt \tag{2}$$

The conditional probability $\mathbb{P}(A\,|H_\emptyset)$, i.e., the probability that no event occurred in $[t, t + dt)$ is given by

$$\mathbb{P}(A\,|H_\emptyset) = 1 - [(1-n)\,\lambda_c(t)\,dt + n\,\lambda_e(t)\,dt] \tag{3}$$

where the factors n and $(1 - n)$ are required in order for the expression to be correct. Above, the probability that a capture occurs when the trap is in state n is set as $(1-n)\,\lambda_c(t)\,dt$ A capture event has a nonzero probability only if the trap is empty, i.e., if $n = 0$. If the trap is full, i.e., if $n = 1$, then this expression correctly yields zero capture probability. Similarly, the probability that an emission occurs when the trap is in state n is set as $n\,\lambda_e(t)\,dt$. And hence, the probability that no event occurs in $[t, t + dt)$ is given by (3).

Since the events $H_\emptyset$, H_c and H_e are disjoint and at least one of them must happen, the law of total probability [44] gives us

$$\begin{aligned}\mathbb{P}(A) = \mathbb{P}(A\,|H_\emptyset)\mathbb{P}(H_\emptyset) + \\ \mathbb{P}(A\,|H_c)\mathbb{P}(H_c) + \mathbb{P}(A\,|H_e)\mathbb{P}(H_e)\end{aligned} \tag{4}$$

If we substitute (1), (2) and (3) in (4), we obtain

$$\begin{aligned}&\mathbb{P}(n, t+dt) = \\ &[1 - ((1-n)\,\lambda_c(t)\,dt + n\,\lambda_e(t)\,dt)]\,\mathbb{P}(n,t) + \\ &\lambda_c(t)\,dt\,\mathbb{P}(n-1,t) + \lambda_e(t)\,dt\,\mathbb{P}(n+1,t)\end{aligned} \tag{5}$$

We rearrange the terms above

$$\begin{aligned}&\frac{\mathbb{P}(n,t+dt) - \mathbb{P}(n,t)}{dt} = \\ &-[(1-n)\,\lambda_c(t) + n\,\lambda_e(t)]\,\mathbb{P}(n,t) + \\ &\lambda_c(t)\,\mathbb{P}(n-1,t) + \lambda_e(t)\,\mathbb{P}(n+1,t)\end{aligned} \tag{6}$$

and take the limit as $dt \to 0$ to arrive at

$$\begin{aligned}\frac{\mathrm{d}\,\mathbb{P}(n,t)}{\mathrm{d}t} = &-[(1-n)\,\lambda_c(t) + n\,\lambda_e(t)]\,\mathbb{P}(n,t) + \\ &\lambda_c(t)\,\mathbb{P}(n-1,t) + \lambda_e(t)\,\mathbb{P}(n+1,t)\end{aligned} \tag{7}$$

The above, called the Chapman-Kolmogorov equation (CKE) [50] in the theory of stochastic processes, known as the Master equation [46] in physics, provides the ultimate, complete stochastic characterization for the two-state Markov chain model for the trap under consideration. If we evaluate (7) for the two states $n = 0$ and $n = 1$, keeping in mind that $\mathbb{P}(n,t)$ is zero for all other n, we obtain

$$\frac{\mathrm{d}\,\mathbb{P}(0,t)}{\mathrm{d}t} \;=\; -\lambda_c(t)\,\mathbb{P}(0,t) + \lambda_e(t)\,\mathbb{P}(1,t) \tag{8}$$

$$\frac{\mathrm{d}\,\mathbb{P}(1,t)}{\mathrm{d}t} \;=\; -\lambda_e(t)\,\mathbb{P}(1,t) + \lambda_c(t)\,\mathbb{P}(0,t) \tag{9}$$

which is a system of two simple linear differential equations with time-varying coefficients that can be easily solved for the state probabilities $\mathbb{P}(0,t)$ and $\mathbb{P}(1,t)$ if the time-varying capture and emission rates $\lambda_c(t)$ and $\lambda_e(t)$ are available. We note that (8) and (9) yield

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbb{P}(0,t) + \mathbb{P}(1,t)] = 0 \tag{10}$$

which also follows from the fact that the total probability of being in state 0 or 1 is always 1, i.e.,

$$\mathbb{P}(0,t) + \mathbb{P}(1,t) = 1. \tag{11}$$

C. Stochastic Differential Equation for a Trap

The Markov chain model for a trap described above can be regarded as a continuous-time, discrete-space stochastic model. As we discuss later, this model could become too costly (computationally) when a large number of traps and a circuit are jointly analyzed. We now derive an approximate, coarse-grained, continuous-time, continuous-state stochastic model for a trap in the form of a stochastic differential equation (SDE) [46], which can offer considerable computational cost savings in the joint analysis of a circuit and a large number of traps.

We consider a time interval $[t, t+\tau)$ and assume that the capture and emission rates $\lambda_c(t)$ and $\lambda_e(t)$ are constant over this interval. Then, the trap occupancy function $N(t)$ can be (approximately) expressed as follows

$$N(t+\tau) \approx N(t) - \mathcal{P}_e(N(t),\tau) + \mathcal{P}_c(N(t),\tau) \tag{12}$$

where $\mathcal{P}_e(N(t),\tau)$ and $\mathcal{P}_c(N(t),\tau)$ represent the (random) number of emission and capture events that occur in the time interval $[t, t+\tau)$. With the above approximation, which is known as the *tau-leaping* condition [43], [44], we are allowing a multiple number of emission and capture events in $[t, t+\tau)$. As such, the resulting $N(t+\tau)$ is not guaranteed to be 0 or 1, even if $N(t)$ satisfies this condition. This is due to the approximate nature of the leap condition that led us to (12). In order to make (12) more concrete, we need to precisely characterize the random variables $\mathcal{P}_e(N(t),\tau)$ and $\mathcal{P}_c(N(t),\tau)$. It follows that these random variables can be approximated as simple Poisson (counting) random variables with parameters $N(t)\,\lambda_e(t)\,\tau$ and $(1-N(t))\,\lambda_c(t)\,\tau$ respectively. The factors $N(t)$ and $(1-N(t))$ have been introduced into these expressions based on similar reasoning as for the ones in (3). However, with the continuous approximation we are developing here, $N(t)$ will take all values between 0 or 1 (and possibly outside), perhaps representing a *fractionally full* trap. As such, with the $N(t)$ factor appearing in the parameter for the Poisson random variable $\mathcal{P}_e(N(t),\tau)$ that represents the number of emission events, having i emissions in $[t, t+\tau)$ will have a probability of $e^{-N(t)\,\lambda_e(t)\,\tau}\,N(t)\,\lambda_e(t)\,\tau/i!$. When $N(t)=0$, number of emissions will be zero with probability one, whereas for larger values of $N(t)$, higher number of emissions will become more probable.

The approximate equation in (12) can in fact be directly used in simulating (in a Monte Carlo manner) a trap with an appropriate choice (most likely, adaptive) for the time step τ [43], [44]. However, (12) is not yet a truly continuous-space model in a differential equation form. $N(t)$ in (12) is still integer valued, though not restricted to be 0 or 1 anymore. We proceed further as follows. The Poisson random variables $\mathcal{P}_e(N(t),\tau)$ and $\mathcal{P}_c(N(t),\tau)$ have means (and also variances) equal to $N(t)\,\lambda_e(t)\,\tau$ and $(1-N(t))\,\lambda_c(t)\,\tau$ respectively. The increment term in (12) is composed of the difference of two Poisson random variables which has a probability distribution known as the Skellam distribution [51]. This distribution has as its mean, the difference of the means of the Poisson random variables and as its variance, the sum of them. For large mean values, the Skellam distribution can be approximated by a Gaussian distribution with the same mean and variance. We approximate

$$\begin{aligned} N(t+\tau) \approx\ & N(t) + (1-N(t))\,\lambda_c(t)\,\tau - N(t)\,\lambda_e(t)\,\tau \\ & + \sqrt{|(1-N(t))\,\lambda_c(t) + N(t)\,\lambda_e(t)|\,\tau}\,\mathcal{N}(0,1) \end{aligned} \tag{13}$$

where $\mathcal{N}(0,1)$ is a zero mean Gaussian random variable with variance equal to one. With the Gaussian approximation, $N(t)$ is now continuous valued. Next, we recognize (13) above as the Euler discretization of the following SDE

$$\begin{aligned} dN(t) =\ & (1-N(t))\,\lambda_c(t)\,dt - N(t)\,\lambda_e(t)\,dt \\ & + \sqrt{|(1-N(t))\,\lambda_c(t) + N(t)\,\lambda_e(t)|}\,dW(t) \end{aligned} \tag{14}$$

where $W(t)$ is the Wiener processes [50]. In physics, an SDE in the form above is known as a Langevin equation and a Wiener process as a Brownian motion. The differential of the Wiener processes, $dW(t)$ in (14), is simply a white noise process. SDEs are commonly expressed in differential form as in (14) with differentials of Wiener processes as opposed to a regular differential equation form with white noise forcing, since white noise is, mathematically, not well defined.

The SDE in (14) is a linear equation, but the noise source (the Wiener process) is modulated with a state ($N(t)$) dependent term. This SDE serves as a nonstationary stochastic, continuous-time, continuous-space differential equation model for a trap. It can be solved numerically (in a Monte Carlo manner) using the Euler discretization in (13). It can also be used as an analytical tool in performing (semi) analytical joint analysis for a circuit and a number of traps. Even though the SDE model was derived after a number of approximations starting with the Markov chain model and the CKE in (7), it can be used to produce the correct (and exact) first and second-order probabilistic characteristics for a trap, exactly the same as one would have obtained based on the CKE, which we state without proof. We note that in the doubly stochastic case, where there is also a coupling from the trap states to the event rates (λ_e, λ_c) the above statement no longer holds and the first and second-order statistics can only be approximated. Here, a first and second-order characterization refers to the mean behavior and all of the second-order moments, i.e., variance, autocorrelation function and spectral density.

III. JOINT ANALYSIS OF A CIRCUIT AND TRAPS

A. Preliminaries and Trap Model in PSP MOSFET model

We have so far developed nonstationary stochastic models for the occupancy function for a single trap. Since the traps are assumed to be stochastically independent, the generalization of the single trap model to a collection of traps in a single transistor or many transistors is trivial. On the other hand, a two-way link between the trap occupancy function $N(t)$ and the currents and voltages of a transistor needs to be established, i.e., we need models on how the currents in a MOSFET are affected by a certain trap in the gate oxide and how the capture and emission rates are a function of the voltages across the transistor.

There seems to be an unsettled controversy in the low-frequency noise modeling literature as to whether charge carrier traps influence the current in a transistor by changing the *number*, or by modulating the *mobility*, of free charge carriers [16]. In this work, we subscribe to the former "school of thought" and model the effect of a captured charge carrier in the channel of the transistor by a change in its flatband voltage, ΔV_{FB}, [52]

$$\Delta V_{FB} = \frac{q}{C_{ox}\,W_{eff}\,L_{eff}} \tag{15}$$

This change has a direct impact on the threshold voltage of the transistor and thus, on its charge and current values. The V_{FB} shift for a single charge carrier can be multiplied with the fractional trap occupancy function in (14) and incorporated into the continuous state model as well. Moreover, the trapping events of charge carriers can be modeled with the classical Shockley-Read-Hall (SRH) theory of recombination [53]. Based on this model, the capture and emission time constants, $\tau_c = 1/\lambda_c$ and $\tau_e = 1/\lambda_e$, of the random process are estimated as

$$\tau_c = \frac{1}{c\,\bar{v}\,\bar{\sigma}} \quad , \quad \tau_e = \tau_c\,e^{-(E_T\,-\,E_F)/kT} \tag{16}$$

The key variable in these equations is c, the carrier concentration at the trap location. The remaining variables are defined as follows: $\bar{v}$ is the average velocity of charge carriers, $\bar{\sigma}$ is a constant called the capture cross section of the trap, E_T is the trap energy level and E_F is the Fermi level.

In order to include (15) and (16) in our simulation framework, we employ the surface potential based compact MOSFET model, PSP [54]. PSP calculates the current trough the device by determining the position dependent electric potential at the channel of the transistor. The charge density at the same location is then obtained from this value and translated into the drain current. During this process, the carrier concentration variable, c, in (16) can be extracted from the charge density information. The integration of the flatband voltage change (15) into the compact model can be accomplished easily by modifying the instance parameter `VFB_i`.

We note that the methodology we describe next is not limited to this flatband voltage model and can use any mapping between the trap occupancy function and the current fluctuation in a transistor. This mapping need not be memoryless and can possibly incorporate nontrivial dynamics. The same principle applies to the relationship between the voltages across the device and the capture/emission rates of traps.

The KCL and KVL equations for a circuit, using a Modified Nodal Analysis (MNA) formulation, can be written as a system of Differential-Algebraic Equations (DAEs)

$$\frac{\mathrm{d}\,\mathbf{q}(\mathbf{x})}{\mathrm{d}t} + \mathbf{f}(\mathbf{x}) + \mathbf{b}(t) = 0 \tag{17}$$

where $\mathbf{x} \in \Re^M$ is a vector of M circuit unknowns such as node voltages, inductor and voltage source currents, and the nonlinear vector functions $\mathbf{f} : \Re^M \to \Re^M$ and $\mathbf{q} : \Re^M \to \Re^M$ represent the nonlinear memoryless components and the energy storing elements in the circuit. The vector $\mathbf{b}(t)$ represents the external and bias inputs to the circuit. For non-autonomous circuits, $\mathbf{b}(t)$ changes with time t, for autonomous oscillator circuits, $\mathbf{b}(t) = \mathbf{b}$ consists of only the constant bias inputs. In order to capture the impact of traps on the circuit, we modify (17) as follows

$$\frac{\mathrm{d}\,\mathbf{q}(\mathbf{x},\mathbf{N})}{\mathrm{d}t} + \mathbf{f}(\mathbf{x},\mathbf{N}) + \mathbf{b}(t) = 0 \tag{18}$$

where $\mathbf{N} \in \Re^P$ is vector of P trap occupancy functions, and $\mathbf{q}(\mathbf{x},\mathbf{N})$ and $\mathbf{f}(\mathbf{x},\mathbf{N})$ represent the impact of the traps on the KCL and KVL equations possibly in the form of a change in the threshold voltages of the transistors.

In order to have a complete description for the circuit with the traps, we also need a model that describes the dynamics of the trap occupancy function vector $\mathbf{N}(t)$ in addition to (18). For this, we use the Langevin model that was derived in Section II-C in the form

$$\begin{aligned} dN_i(t) &= (1 - N_i(t))\,\lambda_{c,i}(\mathbf{x},t)\,dt - N_i(t)\,\lambda_{e,i}(\mathbf{x},t)\,dt \\ &\quad + \sqrt{|(1 - N_i(t))\,\lambda_{c,i}(\mathbf{x},t) + N_i(t)\,\lambda_{e,i}(\mathbf{x},t)|}\,dW_i(t) \end{aligned} \tag{19}$$

Above, $dW_i(t)\ i = 1,\ldots,P$ represent P independent Gaussian white noise sources. The Langevin equations for different traps are not directly coupled with each other, but coupled with the rest of the circuit equations in (18).

The ultimate model for a trap is the stochastic automaton, i.e., discrete-space Markov chain model that was described in Section II-B. In this case, the dynamics of each entry $N_i(t)$ of $\mathbf{N}(t)$ is governed by an independent, two-state Markov chain fully specified by the transition (i.e., capture and emission) rates $\lambda_{c,i}(\mathbf{x},t)$ and $\lambda_{e,i}(\mathbf{x},t)$. However, we note that there is a two-way coupling between the dynamics of $N_i(t)$ and the circuit variables in $\mathbf{x}$, resulting in a doubly stochastic nature. As such, the Markov chains for the traps need to be jointly simulated/analyzed with the rest of the circuit equations in (18) [8].

We next discuss two joint analysis and simulation paradigms for a circuit and a number of traps.

B. Noise Simulation based on Markov Chain Trap Model

For the most detailed and accurate analysis of a nonlinear circuit with time-varying bias conditions and a number of traps associated with it, the discrete-space, two-state Markov chain models for the traps and the nonlinear DAEs of the circuit are jointly simulated in the time-domain in a Monte Carlo manner. Well established numerical methods for solving nonlinear DAEs in the time domain [55] and simulation techniques for discrete stochastic Markov models [43] are available.

Due to the two-way coupling between the circuit and the traps, the simulation techniques mentioned above need to be properly interfaced with each other. More specifically, the simulation of the discrete-space Markov chain trap model generates a number of events in time that result in jumps in the trap occupancy functions and hence the currents in the MOSFET transistors based on (15). The numerical DAE solver needs to be synchronized with these jump events. Moreover, the state transition rates (capture and emission rates) of the Markov chains depend on the circuit variables that are being computed with the DAE solver. Thus, the jump event times with which the DAE solver needs to be synchronized are not only not known ahead of time but also depend on the outcome of the DAE solver itself.

In dealing with this interfacing problem, one method is to set up integral equations involving the DAE-state dependent Markov transition rates which implicitly determine the timing of the jump events [56], [57], [45]. These equations then need to be solved in an iterative manner coupled with the time-step control mechanism of the DAE solver. The integral equations here can be easily converted to differential equations and appended to the list of the DAEs for the circuit, and a DAE solver that can handle implicit events [58] may be used. With this technique, however, one may incur considerable overhead in the DAE solver by iteratively solving for the jump event times. Alternatively, based on a useful property of Poisson processes, called *Poisson thinning* [59], and its generalization to continuous-time Markov chains, called *uniformization* [60], [61], one can generate jump event times which are known ahead of time. In the thinning/uniformization technique, one uses a homogeneous Poisson process with a known, constant rate in order to generate a number of potential jump event times for the non-homogeneous Markov chains with time-varying transition rates that model the traps. Then, these jump events are selected (sub-sampled) in a probabilistic manner and partitioned among the transitions of the Markov chains. With this technique, a small fraction of the potential jump events may eventually get wasted, but the DAE solver no longer needs to iteratively compute the timing of the implicit jump events. Instead, the time-step control of the DAE solver is simply forced to place a time point at all of the jump event times that are known ahead of time. Moreover, one can avoid wasting jump events based on the use of a further generalization of Poisson thinning, called *adaptive uniformization* [61]. A hybrid simulation technique similar to the one we describe here was proposed in [62] for the simulation of stochastic chemical reaction dynamics with mixed differential equation and discrete Markov models. A similar technique was applied to the simulation of RTS noise in SRAMs and DRAMs in [8].

C. Noise Simulation based on Langevin Trap Model

The hybrid Monte Carlo simulation scheme we described in Section III-B that is based on discrete Markov models for the traps will

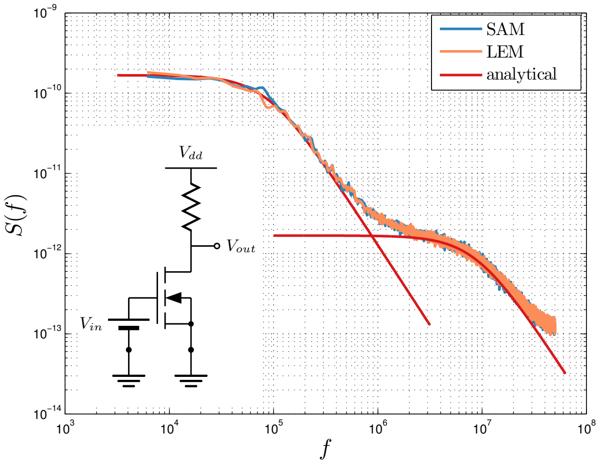

Fig. 1. Power spectral density at the output of a common source amplifier at a fixed bias point. The n-channel transistor contains two oxide traps with different transition rates.

indeed produce the most accurate results as there are no approximations involved. On the other hand, when simulating a nonlinear circuit with a large number of transistors and hence a very large number of traps, the computational cost of this scheme can become prohibitive due to very high density of jump events. In this case, the DAE solver will be forced to take very small time steps that are essentially dictated by the need to pause at all of the jump events. This limits the use of this accurate simulation scheme with circuits that have at most tens of transistors.

A more computationally efficient joint simulation scheme can be developed based on the approximate, continuous-space, Langevin SDE model that was derived in Section II-C, as one is no longer required to deal with discrete jump events. In this technique, the circuit DAEs in (18) are numerically solved along with one additional (stochastic) differential equation as in (19) per trap. However, due to the white noise sources (with state-dependent intensities) in (19), the joint solution of these nonlinear differential equations needs to be performed with specialized numerical integration techniques for SDEs [63], [64]. When the SDE model in (19) is used for the traps and the combined circuit DAEs and the trap SDEs are solved numerically in a Monte Carlo manner (to generate sample paths for the circuit variables and the trap occupancy functions), the only approximations done are the ones we have stated in Section II-C in deriving the Langevin model from the discrete Markov model.

IV. RESULTS

A. *Implementation Notes and* CIRSIUM

The fundamental differences between the nonstationary noise analysis methods we have developed in this paper and the existing low-frequency noise modeling paradigms necessitated the development of a new simulation tool. The implementation of the stochastic automaton method (SAM) and the Langevin equation method (LEM) requires a compact MOSFET model, extended to include a nonstationary description of oxide traps, as well as a modular interface to a circuit equation generation core for the solution of equation (18) together with (19). To this end, we have developed CIRSIUM, a new CIRcuit SImulator Under Matlab® [65]. CIRSIUM has an object oriented design, and provides a flexible and modular framework that enables the rapid development of new device models and circuit analysis techniques. The integration of the SUNDIALS suite [58] with the simulator core and the use of sparse data structures allows for a fast and accurate solution of dynamical circuit equations. Moreover, a conversion module making use of ADMS [66] has been developed for the translation of the PSP compact MOSFET model from the Verilog-A hardware description language, which also automatically generates the code for the computation of Jacobians. This module can be used for the integration of other Verilog-A models into CIRSIUM as well.

The two analysis techniques, SAM and LEM, require different stochastic simulation methods. For the simulation of the discrete events in SAM, we employ the adaptive uniformization technique [61] to determine the occurrence points of capture/emission events ahead of time. At the time of the i^{th} event t_i, the time to the next event is determined by generating a sample of an exponential random variable, $\Delta T_i \sim \text{Exp}(\lambda)$, where the rate parameter λ is chosen such that $\lambda > \sum_{k=1}^{P} \lambda_k(t_i)$. Here, $\lambda_k(t)$ is the voltage dependent event rate of the k^{th} trap. The differential equation solver is then forced to put a time point at $t = t_i + \Delta t_i$ and another random variable determines which trap will make a transition, where the likelihood of a transition is governed by the rates $\lambda_k(t_i + \Delta t_i)$ [67].

Contrary to the discrete nature of SAM, the Langevin model uses a continuous random process to account for the noise generated by the stochastic behavior of the oxide traps. Thus, once the routines for the evaluation of the coupled differential equations (17) and (18) are in place, numerical SDE solution techniques can be used to simulate the effects of the traps on the circuit. Efficient implementations of these techniques require the Jacobians of voltage and current variables with respect to the trap states and vice-versa, which were included in the extended PSP model as well.

B. *Common-Source Amplifier*

As a first example, we present the noise analysis results for a common source amplifier. This circuit consists of a single n-channel transistor and a resistor (Figure 1). The transistor is biased with a fixed gate voltage, V_{in}, and hence, the only dynamical behavior in the circuit is caused by the activity of oxide traps. A single such trap is expected to cause a noise voltage component at the output of the circuit with a Lorentzian power spectral density (PSD)

$$S(f) = \frac{2\Delta V^2 \lambda}{4\lambda^2 + (2\pi f)^2} \tag{20}$$

where ΔV is the change in the output voltage due to a capture/emission event. Multiple traps with different transition rates result in a PSD in the shape of a superposition of multiple Lorentzians as confirmed by measurements [68].

Using CIRSIUM, we have simulated this common source amplifier circuit with SAM as well as LEM. We have placed two interface traps in the transistor with transition rates adjusted to be approximately two decades apart from each other at the bias point. Figure 1 compares the results obtained with the two simulation methods for the noise voltage power spectra at the output of the circuit. The power spectra were estimated from the time series data accumulated by running the SAM and LEM simulations long enough to capture the low frequency spectral components. As expected, we obtain a monotonically decreasing power spectrum with two "corners". It is not surprising that the SAM algorithm which relies on discrete switching events as noise sources closely matches the analytical results in (20) (smooth, red curves). The noise model in the LEM algorithm, on the other hand, has a structure that does not exhibit any resemblance to a random telegraph signal but, nevertheless, its accuracy in predicting the impact of the oxide traps on the output voltage noise spectrum is as good as the discrete algorithm based on stochastic automatons.

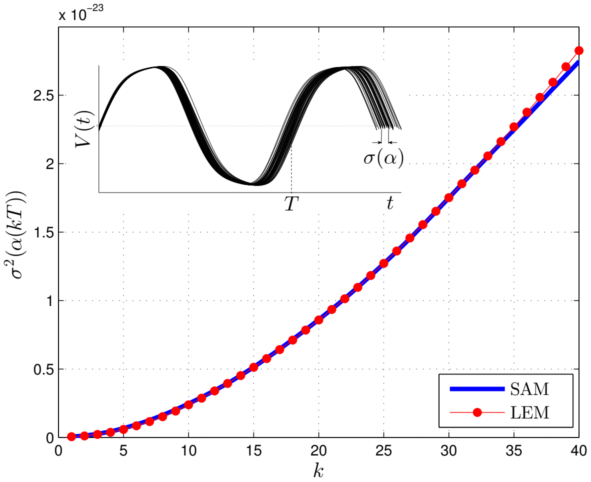

Fig. 2. Phase jitter variance of a three-stage ring-oscillator with a total of twelve non-stationary traps. The inlay shows a snippet from 100 waveforms generated by our simulations, drawn on top of each other.

C. Phase Jitter in a Ring-Oscillator

As a second application of our nonstationary, low-frequency noise analysis framework, we analyse the effects of low-frequency noise on the accumulated phase jitter of a ring-oscillator. The ring-oscillator is an autonomous oscillator circuit, and hence, it exhibits phase fluctuations with increasing variance. Contrary to the common source amplifier, this circuit has truly nonstationary noise characteristics due to the nature of these phase fluctuations [20]. Low frequency noise in oscillators causes fluctuations in the zero-crossing times of the periodic waveforms known as phase or timing jitter (shown in the inlay of Figure 2). For instance, in an n-channel transistor, the threshold voltage increases due to the capture of an electron by a trap, resulting in a delay in the discharging process of the next stage which subsequently causes timing jitter.

Here, we analyse a ring-oscillator consisting of three CMOS inverter stages connected in a closed loop chain. Each transistor in the inverters contains two oxide traps with different capture cross sections and therefore with differing voltage dependent transition rates. In order to characterize the variation in the phase evolution of the circuit, we consider the output voltage of the first inverter

$$V(t) = x(t + \alpha(t)) + y(t)$$

where $x(t)$ is the periodic waveform of the unperturbed oscillator, $y(t)$ represents the variation in the amplitude of this signal and $\alpha(t)$ is the phase deviation [20]. We determine $\alpha(t)$ for each multiple of the oscillation period, kT, by calculating the crossing times of the waveforms through the level corresponding to the half of the power supply voltage. In a Monte Carlo manner, we proceed to build ensembles of a large number of simulations and determine the variances of the random variables $\alpha(kT)$.

Figure 2 compares the results obtained from simulations with SAM and LEM. As in the previous example, the Langevin model closely matches the discrete event model. Moreover, the results have the correct functional form as predicted by the theory of phase noise in oscillators with colored noise sources [27].

D. Discussion

We have demonstrated the accuracy of the newly developed Langevin equation model for low-frequency noise analysis on two example circuits. The first example, the common source amplifier, confirmed that our model accurately predicts the power spectral density of noisy signals in transistors under constant bias conditions. The second and more interesting example, the phase jitter characterization for a ring-oscillator due to nonstationary, low-frequency noise with multiple traps that exhibit nonstationary characteristics, belongs in a realm where currently there are no theoretical results. This example demonstrates the ability of LEM to cover a large area of more complicated low-frequency noise behavior. In addition to the speed increase in such simulations provided by LEM over SAM, our methodology allows for further development of robust, efficient and possibly non Monte Carlo methods in nonstationary, low-frequency noise analysis.

V. CONCLUSIONS

The RTS and low frequency noise simulation techniques we have described in this paper are of the stochastic, Monte Carlo, simulation type, which usually become quite inefficient in analyzing large and complex systems. However, the Langevin trap model proposed in this paper can be incorporated into the non Monte Carlo noise analysis algorithms that are currently implemented in circuit simulators for RF circuits and phase noise analysis of oscillators [15], [24], [23], [20]. This forms part of our current and future work.

In this paper, we focused on how one can benefit from the ion channel noise modeling and stochastic chemical kinetics literature in developing models and analysis techniques for RTS and low-frequency noise in electronic circuits. However, we believe that the non Monte Carlo noise analysis techniques that have been developed recently for electronic circuits can be put to use in large-scale analysis of noise in the nervous system [32], [69]. We are currently also pursuing this line of work that emphasizes a two-sided cross-fertilization between the two disciplines.

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